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# Pricing of Credit Risk and Credit Risk Derivatives: *From Theory to Implementation*

A dissertation submitted in partial fulfilment of the  
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I declare that this dissertation is my own, unaided work. I know the meaning of plagiarism and declare that all of the work in the document, save for that which is properly acknowledged, is my own. It is being submitted for the Degree of Master of Science to the University of Cape Town, Cape Town. It has not been submitted before for any degree or examination to any other University.

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Signature of Candidate

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Date

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Neville Sewnath

November, 2007

## Abstract

The sophistication of financial products has largely changed the nature of the way banks do business. The proliferation of financial derivatives combined with the complexity of the inherent payoffs of structured portfolios has induced banks to develop new methodologies to assess and manage the credit risk arising from the different aspects of their business processes. In chapter 1 of this study we review and discuss the mathematical credit risk models and credit risk derivative models in literature that are primarily characterized by Brownian motion dynamics although it is widely accepted that observable credit risk data exhibits somewhat different statistical dynamics than that implied by theoretical models. In particular, we emphasize the pricing of credit risk as an application of contingent claim analysis and we consider an example of a default risky claim as an application of pricing a credit risk derivative.

In chapter 2 we review a sample of the basic concepts and assumptions in mathematical finance that support the theory we present in subsequent chapters. We observe that some aspects of credit risk modelling is developed from the technology of interest rate modelling and as such we present a concise overview of the relevant theory of interest rate models and bond markets that is applicable to our models. Finally, as an application of martingale pricing theory we derive the Black-Scholes equity option pricing formula.

Chapter 3 gives an exposition of the category of credit risk models widely known as structural models that is commonly associated with Merton. First, we give an in depth review of Merton's firm value model as an application of contingent claim analysis. Second, we generalize the classical Merton model to show a new class of models termed first passage time models. We postulate a general first passage time model associated with stochastic interest rates and apply the Briys and de Varenne approach to derive the fundamental pricing equation of a zero coupon defaultable bond. In addition, we solve the fundamental pricing equation to show a closed-form formula for the price of a default risky zero coupon bond.

The basis of the discussion in chapter 4 is the second category of credit risk models widely known as reduced form models. In particular, we focus our attention on the intensity based approach to credit risk modelling. In contrast to the structural model approach, where the default event is formulated through economic arguments, the intensity based approach characterizes the default time as an entirely random time with Markov dynamics. Notably, this approach does not proffer an intuitive economic explanation to characterize the relationship between firm value and default of the firm. From a theoretical perspective, we show that intensity based models possess the memoryless property and that default time is modelled as a first jump time of a homogeneous Poisson process. In addition, we define a Cox process in which the intensity function is allowed to have stochastic dynamics. We conclude by showing two methods, the first based on a Poisson process and the second based on a Cox process, to derive an identical pricing formula for

a default risky contingent claim.

We continue with reduced-form models in chapter 5. Generally, in reduced-form models the pricing of defaultable contingent claims specifies the recovery rate as an exogenous quantity to be either constant or a quota of the corporate's bond obligation at the instant of default. However, the expanding literature base on the cross-sectional effects that influence the dynamics of credit risk intuit that systematic factors affect both the probability of default and the loss quota at default. We show an application that encompasses this empirical feature in an intensity-based framework.

The second part of this dissertation is devoted to the pricing of credit risk derivatives and this forms the focus of our study in chapter 6. In particular, we discuss the basic structure of three credit risk derivative instruments, a total return swap, a credit default swap and a credit spread option. We present a computer implementation of the Das and Sundaram credit risk derivative model to derive the numerical value for a credit spread option as an application in the reduced form model framework. In the last section, we derive the pricing formula for a credit risky put option with one sided counterparty risk postulated in the structural model framework.

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# Chapter 1

## Introduction

It is commonly accepted that fixed income markets are the largest capital markets worldwide and attributable to capital markets are financial risks. An intuitive interpretation of financial risks can be deemed to refer to the random adverse movements in the market value of some financial instrument, for example, a bond, a share, or a portfolio of financial assets. If an investor holds a corporate bond in the fixed income market, the associated financial risks with the bond are characterized by market risk and credit risk. Both market risk and credit risk are not unrelated since changes in either risk can be linked to changes in prices of financial instruments. On the one hand, market risk is generally considered to be related to movements in the repurchase interest rate, foreign exchange rate, commodity prices and prices of financial instruments. On the other hand, credit risk in its general form evolves from the risk that the obligor of a fixed income security does not fulfil his obligations and hence defaults. Ideally, market risk and credit risk should be addressed jointly because, in the main, a default event is contingent on adverse changes in asset prices and interest rates. Nonetheless, given the importance of market risk in all fixed income markets, the first part of this dissertation is primarily concerned with the pricing of credit risk.

In the present global environment, particularly, in most developed financial markets, and increasingly in emerging financial markets, derivative instruments can be used to manage the different forms of market risk on a firm's balance sheet. The various forms of market risk include share price risk, interest rate risk, foreign exchange risk and a asset price risk. Up until a few years ago not many instruments could adequately help manage and hedge credit risk. However, increased trading in financial instruments subject to credit risk has led to the advent of credit risk derivatives, instruments that are designed to partially or completely hedge the credit risk in a financial transaction. Credit risk derivatives have widely become the derivative's market new frontier and by nature is structured to provide an efficient means of hedging or even acquiring credit risk. In the second part of this dissertation we will characterize various credit risk derivative structures and postulate models to price credit risk derivatives.

## 1.1 Motivation

Financial institutions such as banks hold typically thousands of financial positions. Credit risk, or more commonly the risk of default, is a major financial risk for banks and other financial institutions committed to some financial position. More precisely, credit risk sufficiently requires diligent management from the financial institution in terms of allocation and diversification, and the measurement of exposure of a portfolio and its components to this particular source of risk. Management has increasingly diverted more attention to this sector of the business because of the proliferation of financial derivatives and the complexity of the inherent payoffs of structured portfolios. This phenomenon is essentially driven by globalization that introduces new entrants into the market that have a lower credit standing. Indeed, this new paradigm has spurred the credit term structure as the primary input to many credit portfolio management systems.

In simple terms credit risk is defined as the risk that a counterparty to a financial contract cannot completely meet his financial obligations because the amount of liabilities exceeds the amount of funds available. In the earlier example of a corporate bond in capital markets this usually implies that the obligor has defaulted on a promised payment on its debt securities. Intuitively, this means that the underlying analysis of credit risk is contingent on the modelling of a pre-defined credit event with respect to an obligor or counterparty.

In general, there are three types credit risk, that is, default risk, downgrade risk and credit spread risk. Default risk is the risk that an obligor of a bond, say a corporation, will not be able to fulfil its obligations (coupon, principal) at maturity of the obligation. Default risk can be complete in that no amount of the bond obligation is recovered. Downgrade risk is the risk that a credit rating agency like Standard and Poors will lower the credit rating for an obligor based on a perceived lower earning capacity. Finally, credit spread risk is the risk that the spread over a reference rate, say the yield on a government bond, will increase for the outstanding bond obligation over the time to maturity. These are the primary determinants that financial institutions are increasingly measuring and managing the risk from credit exposures inherent in their portfolios. While it is a well-known fact that possible default of a counterparty on an agreed upon financial position is centuries old, modern methods and models have been developed in the last few years to assess and manage this risk. Some examples of these advanced models are shown in this dissertation.

## 1.2 Objectives

During the recent past there has been a renewed interest in the field of credit risk research that has spurred a proliferation of theoretical models. On the one hand, a vast majority of research continue to expand and introduce real world phenomena into the traditional credit risk models for pricing corporate bonds. On the other hand, the

analysis of credit risk, at least from a mathematical point of view, has seen the advent of new fields of research that encompass the application of actuarial mathematics models to areas like the modelling of default and accounting information or continuous time corporate finance. Notably, this is, in general, a new frontier in mathematical finance that has applications to adequately and efficiently build credit risk models to meet the requirements of the new Capital Accord (2004) [11]. Although the body of knowledge of credit risk analysis is beyond the scope of this dissertation, we, however, derive utility from the concepts and intuition that stimulates this field of research. We aim to demonstrate an exposition that builds on the strengths and diversity of previous models and, most importantly, postulate a framework for fair value contingent claims that explains market phenomena realistically. More specifically this dissertation prices credit risk and credit risk derivatives in the following context:

- The pricing of credit risk in the structural model framework. We aim to value credit risky bonds and to show that these bonds are valued cheaper relative to their risk-free counterparts. We show that credit risk is primarily characterized by default probabilities or, put differently, the probability that an obligor may default on his obligations and this loss is reflected in the distribution of credit risky bond prices. First, we model this valuation problem in the classic Merton (1974) [94] firm value framework. Second, we extend the Merton (1974) [94] model, to approximate real world phenomena, to the first-passage time framework that was initially postulated by Black and Cox (1976) [18]. In particular, we introduce stochastic interest rates and derive closed form solutions under various assumptions of the Briys and de Varenne (1997) [23] approach.
- The pricing of credit risk in the reduced form framework. Reduced-form models emphasize the unpredictable dynamics of a credit event and as such distinguishes itself from the inherent limitations of structural models. Here we model the default time as the stopping time of an exogenously specified hazard rate process. First, we define the hazard rate process as a Poisson process and value a credit risky bond given the default event is the first jump of a Poisson distribution. Second, we define a Cox process to be a doubly stochastic process, or put differently, a Poisson process is said to have a constant intensity and a Cox process is defined to have a stochastic intensity. With the Cox process we show a second method to value a credit risky bond in a risk-neutral valuation framework.
- The pricing of credit risk derivatives. A credit risk derivative is essentially a financial derivative with credit risk as the underlying. First, we briefly describe the structures of three basic credit risk derivatives: total return swap, credit default swap and a credit spread option. Next, we price a credit spread option based on the Das and Sundaram (2000) [34] credit risk derivative model. Lastly, we consider credit risk derivatives with one-sided counterparty default risk. This means that the event that a counterparty can default on his obligation to honour the derivative contract can affect the value of a credit risk derivative. We illustrate this by deriving a closed-form formula for a credit risky put option in the firm value framework.

Needless to say that the main objective of this dissertation is to postulate quantitative models that price and hedge credit risk that are consistent with the assumption of no arbitrage in the structural model framework and the reduced form model framework.

## 1.3 Structure

If we return to our example of a corporate bond investor in capital markets we can say that the single most important state variable that an investor would be interested in is the estimates and properties of default probabilities, either actual or risk-neutral, so that they can gain insight to the likelihood of both the default event and the term structure of the default probabilities. Consequently, from a theoretical perspective we have observed the emergence of two distinct classes of quantitative credit risk models that are commonly known as structural models and reduced form models, respectively.

### 1.3.1 Fundamentals of Credit Risk Modelling

We introduce in Chapter 2 the main ideas of option pricing theory and we develop this theory in the case when asset prices are set in a continuous time economy. Generally, two concepts associated with option pricing are replication and arbitrage. We aim to give the key underlying concepts and definitions that support the idea of replication, or more generally a trading strategy that results in a payoff equal to the value of a contingent claim. In practice, the intuition of first principles is often eclipsed by the application of theorems or formulae when performing calculations. It is important, however, to remind ourselves that the pricing of contingent claims can be accomplished by trading in other assets in a complete economy. In our economy we postulate that an arbitrage opportunity can be formulated as a trading strategy that will result in a non-negative payoff from a zero investment in an additional quantity of assets given that there is no risk in the transaction. Clearly, in our economy for any fair trading strategy there should not be any opportunities for arbitrage to exist. That is, with the absence of arbitrage condition in place, we immediately get the payoff of a contingent claim equal to the value of the trading strategy that replicates it. Although this may be deemed to be trivial but for our purposes we state this as necessary and establish the conditions for this to exist.

A secondary objective of this dissertation is to develop our credit risk models and credit risk derivative models in a continuous time economy. However, pricing credit risk and credit risk contingent claims requires a model for underlying asset price dynamics in the economy. Although in the real world asset prices are observed to be piecewise constant and undergo discrete random jumps, in the mathematical modelling of the theory of finance we have adopted the convention of modelling asset prices by continuous stochastic processes. In this dissertation we frequently choose a geometric Brownian motion to generate the stochastic dynamics of asset prices in continuous time. Geometric Brownian motion adequately represents the intuitive behaviour of an asset price process.

In most cases we develop our models in a risk neutral economy and not in the real world economy. This implies that we need to adopt a change in measure from the objective probability measure,  $\mathbb{P}$ , to an arbitrary equivalent probability measure,  $\mathbb{Q}$ . This change of measure concept is important in continuous time arbitrage pricing theory because it establishes conditions to consider the economy as complete.

For the most part, chapter 2 gives an exposition of simple ideas and concepts that contribute to the theory of contingent claim analysis. The development of these concepts has a peculiar theoretical mathematical approach but when it gets down to the modelling of contingent claims, in practice, the mathematical theory is a useful pre-step. We put to good use these concepts by presenting the standard and widely accepted risk neutral valuation methodology as originated by Harrison and Kreps (1981) [62]. We conclude this chapter by deriving the famous Black Scholes (1973) [20] option pricing formula and many of the techniques demonstrated in deriving the formula will often times be used throughout this dissertation.

### 1.3.2 Structural Models

In chapter 3 we review the traditional class of credit risk models that is widely known as structural models or alternately firm value models. The structural model concept was first postulated by Merton (1974) [94] and is concerned with modelling and pricing credit risk that is associated with a particular corporate obligor. In this model credit risk is viewed as the risk that an obligor cannot meet his obligations at a pre-defined maturity date since the value of his liabilities exceeds the value of his assets. The key assumption of the Merton (1974) [94] model is that the evolution of the firm value process, as a proxy for the asset price process, follows a diffusion process. The firm value process models the dynamics of the prices of the shares issued by the firm and all debt against the firm value are modelled as contingent claims with the firm value as the underlying. More precisely, the debt of the firm is modelled as a portfolio of a risk-free zero coupon bond and a short put option on the value of the firm. We notice that when modelling with a diffusion process, the evolution of the firm value is not characterized by random jumps. This means that firms do not default unexpectedly. Put differently, the time of default is accessible under a diffusion process and hence the default event is predictable. Consequently, the time of default is defined as the first instant when the value of the firm breaches a specified lower threshold. However, default is only triggered when the value of the firm at the specified maturity date is less than the value of the firm's liabilities. In this basic credit risk model default can only occur at maturity.

The next generation of structural models was pioneered by Black and Cox (1976) [18] where default is triggered at the first instant the value of the firm reaches the default threshold. We briefly review the Black and Cox (1976) [18] model. This extension of the Merton (1974) [94] model is commonly known as first passage time models. First passage time models are characterized by bond indenture provisions that include safety covenants which aim to protect bondholders by allowing them to reorganize or foreclose on the firm if the value of the assets of the firm falls below a pre-specified lower threshold for the

first time. Naturally, there has been improvements on the original Black and Cox (1976) [18] model albeit some models exhibit drawbacks. We propose a general framework for a first passage time model that aims, in some ways, to correct deficiencies of previous models. In particular, our first passage time model has three main features at its core that underscores observable real world phenomena and we characterize them as:

- bankruptcy costs are included at the reorganization of the firm
- we assume interest rates are stochastic and modelled by the one factor Vasicek (1977) [118] model.
- we judiciously choose the default threshold and recovery rate such that at reorganization of the firm the payoff to the bondholders do not exceed the value of the firm.

The main result of this section is that we apply the Briys and de Varenne (1997) [23] approach to derive the price of a zero coupon defaultable bond. In addition, we solve the fundamental pricing equation to get a formula for the price of a default risky zero coupon bond.

## Literature Review

In modern finance theory it is widely accepted that the structural models introduced by Black and Scholes (1973) [20] and Merton (1974) [94] have since become the cornerstone of corporate debt pricing. In their seminal work Black and Scholes (1973) [20] propose the intuitive notion of modelling the capital structure of the firm as derivative securities. Merton (1974) [94] makes this concept precise by postulating an analytical methodology to view corporate debt as a portfolio of a riskfree bond and a short put option written on the assets of the firm. Geske (1977) [54] extended Merton's model by demonstrating that multiple default contingent claims for coupons, junior debt, and safety covenants could be priced as compound derivatives.

The traditional Merton (1974) [94] model has been extended in several ways over the years. The Black and Cox (1976) [18] model allows for safety covenants, subordination provisions and limits on refinancing. In particular, the concept of first passage time models is attributed to Black and Cox (1976) [18] where they introduce a safety covenant modelled as an exogenous, time dependent boundary to solve the problem of default prior to maturity. In contrast to the time-dependent default threshold Brennan and Schwartz (1980) [22] propose a constant default threshold in their pricing model for convertible bonds but, however, this results in a numerical solution for the fundamental pricing formula. In their extension of the first passage time model Kim, Ramaswamy, and Sundaresan (1993) [83] introduce a stochastic riskless interest rate model that follows the Cox, Ingersoll, and Ross (1985) [30] square root process. They show that credit risk is not particularly sensitive to volatility of interest rates but, in fact, has a likelihood to be sensitive to interest rate expectations.

Yet another extension of the Black and Cox (1976) [18] model was postulated by Longstaff and Schwartz (1995) [91] where they derive semi-closed form solutions for the fundamental default risky bond pricing equation in the firm value setting. First, they propose riskfree interest rates that follow the stochastic dynamics of the Vasicek (1977) [118] model and the interest rates can be correlated with the firm value process. Second, they propose an exogenously defined recovery rate that is explicitly independent of the default threshold at the instant of default. This implies that the model does not allow the variation of the recovery rate of a defaultable bond to be linked to the value of the firm at default. Briys and de Varenne (1997) [23] identified this anomaly in the Longstaff and Schwartz (1995) [91] model and postulated a solution that suggests default is triggered upon first passage of the forward firm value.

The first extension of the Black and Cox (1976) [18] first passage time model was introduced by Mason and Bhattacharya (1981) [93] where their model admitted jump processes to characterize the value of the firm. The main feature of this model was that the default time was specified as an inaccessible stopping time. A generalization of jump processes was pioneered by Schönbucher (1998) [111] and Zhou (2001) [121], of which is motivated by the empirical investigation of Jones, Mason, and Rosenfeld (1984) [77], where they show that credit spreads on corporate bonds are too high to be matched by the classic firm value approach.

Over the past few years several new paradigms for measuring and controlling the risk inherent in credit sensitive assets have been conceptualized, developed and marketed as commercial credit risk management models. In particular, some of these models retain the economic appeal of the structural approach and integrate the empirical plausibility to measure and quantify credit risk at both, the portfolio and individual financial investment level. One such model is CreditMetrics<sup>TM</sup> developed by J.P. Morgan (now J.P. Morgan Chase). The CreditMetrics<sup>TM</sup> methodology models the forward distribution of the values of a loan or bond portfolio over an arbitrary chosen forward time horizon, usually one year. The changes in these values are related to the probability of moving from one credit state to another within the chosen time horizon, including default, and combining these individual value distributions to generate a loss distribution for the overall portfolio. CreditMetrics<sup>TM</sup> builds its model from data on ratings and price variations in the liquid bond market and the primarily corporate bond driven credit risk derivative market where financial instruments are actively traded.

A second model that follows Merton's (1974) [94] credit risk methodology insight was developed by KMV Corporation. KMV specializes in credit risk analysis and has built up an extensive historical database to model default probabilities and portfolio loss distribution that results from default and credit rating migration. The KMV framework uses a version of Merton's (1974) [94] structural model to define the *distance to default* (DD) category to which a counterparty belongs. This DD is then mapped to an *expected default frequency* (EDF) from the historical database, which results in an implicit credit rating for the counterparty. In the main, the essential difference between the KMV and



CreditMetrics<sup>TM</sup> methodology is that KMV uses EDF's for each obligor and not the historical transition frequencies produced by rating agencies like Moodys or Standard and Poors as a measure of credit risk. Both the KMV and CreditMetrics<sup>TM</sup> framework for modelling portfolio credit risk is adapted to Merton's (1974) [94] classical structural approach.

### 1.3.3 Reduced Form Models

The basis of our discussion in chapter 4 is the second category of credit risk models widely known as reduced form models. In this dissertation we will only treat the class of reduced form models that are concerned with the modelling of the default time and not the class of models that treat the migration between credit states. In contrast to structural models where we model the relation between the default event and the value of the firm in an explicit manner, in the reduced form approach, we model the default event as the unpredictable first jump of a Poisson process that involves a sudden loss in market value of the financial instrument. In this context, we alternately, refer to reduced form models as the intensity based approach because the default time is always modelled as an inaccessible stopping time and the default event is often formulated in terms of a hazard rate function.

The desirable feature of the intensity based approach is that it offers model tractability. Notably, this approach does not proffer an intuitive economic explanation to characterize the relationship between firm value and default of the firm. Put differently, the hazard rate of default in the intensity based approach is specified as an exogenous process which is stochastic by nature and characterized by Markov dynamics and as such the implication that a firm default is a surprise event is economically imperfectly plausible. However, due to the unpredictable nature of a firm default the intensity based approach is usually more flexible to be calibrated to market data and the parameterized implied credit spreads are economically more plausible.

The cornerstone assumption of intensity based models is the modelling of default time as an entirely random time with Markov dynamics. We show that the exponential distribution adequately emphasizes this assumption by remonstrating that it possesses the memoryless property. From a theoretical perspective, the memoryless property is defined in terms of independent and stationary increments and we show it is exactly this property that explains the intuition of the intensity based approach of equating the default time to the first jump time of a homogeneous Poisson process. A characteristic of the homogeneous Poisson process is that it has a constant intensity rate. We define a non-homogeneous Poisson process to be characterized by a deterministic intensity function. The concept of a time-varying intensity function is expanded to define a Cox Process to be a generalization of the non-homogeneous Poisson process in which the intensity function is allowed to have stochastic dynamics.

The pricing methodology for default risky contingent claims closely approximates that

for default free contingent claims with the differentiating feature being the discount rate. In the risk neutral valuation framework we discount with a default adjusted rate,  $r + \lambda$ , with  $\lambda$  being exactly the intensity. In the intensity based approach we use both the Poisson and Cox process to close the model for pricing default risky contingent claims. We concentrate our exposition on illustrating the identical price for a default risky contingent claim by using two different approaches. First, we state the fundamental results derived by Bielecki and Rutkowski (2000) [13] based on the Poisson process approach. Second, we construct a Cox process as postulated by Lando (1997) [85] and use this to show a second method to derive the price of a default risky contingent claim.

## Literature Review

Over the years research on reduced form models has contributed to an ever growing literature base that range from simple hazard rate models to complex model postulates where the hazard rate can be defined in terms of the recovery rate at default. The earliest approaches of reduced form models were proposed by Ramaswamy and Sundaresan (1986) [105], Litterman and Iben (1991) [90] and, Jarrow and Turnbull (1995) [73]. We can find models that are in a discrete time framework and other models are in a continuous time economy.

A widely known discrete time model was proposed by Jarrow and Turnbull (1995) [73]. Their model is consistent with a zero coupon Treasury bond term structure and a zero coupon corporate bond term structure for a specified credit rating class. They propose stochastic interest rates but define exogenously the processes for the default event and the payoff on the risky debt conditional on the default. The model mimics the foreign exchange mechanism to construct the dollar payoff from a defaultable security to comprise a certain payoff and the stochastic *spot exchange rate*. Their model is set in a risk neutral framework and can be applied to a basket of financial instruments.

In their model Jarrow, Lando, and Turnbull (1997) [72] propose a Markov model for the term structure of credit spreads as an extension of the earlier Jarrow and Turnbull (1995) [73] model. They postulate linking the default process to a discrete state space characterized by credit rating migrations. The credit rating migrations are defined as Markov transitions between rating categories with default defined as the absorbing state. A feature of this model is that it offers a fair deal of flexibility to parameterize the observable economic data and with the appropriate assumptions we can price contingent claims. The base assumption of this approach is that the credit rating is assumed as the indicator of credit worthiness which makes it the crucial variable on which the payoff of credit risk derivatives are contingent. Consequently, this assumption presents a drawback of the model since the rating categories present a fair amount of variation in the credit quality of bonds within each rating category. This supposes that the model does not assume homogeneous discreteness in its structure of credit ranking.

The Duffie and Singleton (1999) [42] model represents the possible basis for a series

of reduced form models. Default is treated as an unpredictable variable modelled by a Poisson process with state dependent variables for the hazard rate and loss in default. In particular, they price contingent claims under the risk neutral measure that discounts the risk-free payoff on the debt by a default adjusted short term rate process computed as the sum of the short term risk-free rate and a factor that represents the default risk premia. The default adjusted short term rate process supposes that we can price contingent claims as if it was risk-free.

In his paper Lando (1998) [85] presents a modelling framework for credit risky securities and credit risk derivatives that encompasses the dependence of the default free term structure of interest rates and the defaultable characteristics of the firm. In particular, Lando (1998) [85] constructs a Cox Process and as an implementation he presents a generalization of the model by Jarrow, Lando, and Turnbull (1997) [72] to allow for stochastic transition intensities between credit rating categories.

### 1.3.4 Credit Risk Derivatives

The advent of credit risk derivatives is reasonably recent in comparison to bond insurance that was introduced approximately 30 years ago and was also designed to have a payoff contingent on a default event. Additionally, letters of credit and surety bonds, also default contingent instruments, have been in use much longer. From a historical perspective, corporates traditionally opted to manage credit risk of commercial contracts by trading in the underlying itself or by buying insurance. With specific conditions in place, securitization of receivables was also an alternative. However, most of these alternatives constituted varying levels of protection associated with significant costs. Given that the estimated notional amount for credit risk derivatives to be somewhere near \$1.6 trillion for the year 2001 [117] to barely approximate one percentage point when compared to the financial derivatives market or cash credit market it is sometimes intriguing to see why so much research is being applied to these new instruments and to assess whether they really do lead to completing financial markets. Notably, many market participants take the view that credit risk derivatives is a new frontier in financial markets that is redefining our banking and regulatory framework and creating new opportunities in banking and insurance.

Credit risk derivatives are over-the-counter derivatives securities the value of which derives, at least in part, from the credit characteristics of the reference financial securities. For example, credit risk derivatives allow investors to trade the risk elements embedded in reference bonds and loans, and to construct synthetically portfolios with specific credit risk profiles. A key feature of credit risk derivatives is that they separate the ownership and management of credit risk of the legal and regulatory requirements of ownership of financial securities. This means that financial intermediaries can preserve their client-customer confidentiality while discreetly managing their credit risk exposure. It is exactly this feature of disaggregating specific aspects of credit risk from other risks that makes for the application of credit risk derivatives to even the most illiquid portfolio

credit exposures enticingly attractive to investors.

Broadly, credit risk derivatives can be specified as three main categories. In this dissertation we review one example from each category. As an example of the category of credit risk derivatives that allow for the complete exchange of risk of a financial security between counterparties we describe a *total return swap* structure. A total return swap is a derivative contract between two counterparties whereby one counterparty (the rate payer) makes periodic fixed or floating rate payments to the second counterparty (the total return payer) and receives from the total return payer the total return, principal and coupon payments net of the differential in the price movements of the reference asset for the period of the contract.

The second category of credit risk derivatives is specified as being explicitly linked to the default event and the payoffs are contingent on the default event. Here default is defined strictly as non compliance to meet a negotiated financial obligation in contrast to a credit migration. A credit default swap is placed in this category and is defined as a derivative contract between two counterparties whereby one counterparty (the protection seller) receives fixed periodic payments from the second counterparty (the protection buyer) in return for making a single contingent payment that recovers losses on a reference asset following the specified default event. Finally, the last category of credit risk derivatives is characterized by the term structure of the credit quality of the reference asset. We include in this category credit spread derivatives that are defined as options linked to a credit spread, that is, the difference between the current yield of a reference asset and that of a benchmark or risk-free security. As much as we like to advance towards industry wide conventions we notice that the credit risk derivative field is still in flux and the above specified categories may not be strictly definitive.

In their paper Das and Sundaram (2000) [34] present a model for credit risk derivatives pricing that is arbitrage free, accommodates path dependence, and handles a range of securities and can be extended to price securities with American features. Their approach directly models the forward rates and the credit spreads in a double binomial modelling framework that can easily be implemented in a lattice structure. The framework is developed in a discrete time HJM (1990) [64] model as the basis for its engineering implementation. In particular, we perform a computer implementation for a credit spread option using Microsoft Excel.

In the last section of this chapter we present a firm value pricing model for derivatives with one sided counterparty credit risk. The credit risk model closely resembles the firm value model developed by Merton (1974) [94]. In particular, we postulate a model that is developed in the Black-Scholes (1973) [20] framework where both the firm value and the value of the underlying asset follow a geometric Brownian motion process under the assumption of constant interest rates and deterministic liabilities. In addition, we derive a closed form pricing formula for a credit risky put option with one sided counterparty risk.

## Literature Review

It is a well known proposition of fact that the concept of credit risk is an age old economic risk in so far as a counterparty default on a negotiated obligation. To hedge against potential losses some types of credit risk derivatives have been in use for a long period but were known under a different name. For example, the concept of general average in marine transport is a centuries old tradition. In comparison relatively more recently default insurance products priced on the original Merton (1974) [94] valuation model represent modern applications of credit risk derivatives.

The current thrust of *Credit Risk Definitions* was published by the International Swaps and Derivatives Association (ISDA, 1999) and is viewed as a bold move to standardizing the terminology in credit risk derivatives contracts. Following minor market ambiguous interpretations of the ISDA guidelines the ISDA Definitions concerning which obligations can be delivered in physically settled contracts in the case of a debt restructuring event were amended in 2001. Nonetheless, the Definitions established an industry wide set of guidelines of important terms such as the range of credit events that could trigger payments or deliveries. In addition, to the robust enforceability and interpretation of the contracts, the Definitions increased flexibility and simplified the documentation and administration processes.

There has been largely a lot more articles published on the pricing of defaultable bonds and derivatives with embedded credit risk in comparison with articles based on the direct pricing of credit risk derivatives. Following the publication of the ISDA guidelines (1992) on credit risk derivatives, the article by Das (1995) [33] is ranked among the earlier research associated with the guidelines. In his paper, Das (1995) [33] presents a contingent claims approach to the pricing of derivatives on the credit risk of corporate debt. The model basically shows that in an asset based framework credit risk derivatives are the expected forward values of put options on defaultable bonds with a credit level adjusted exercise price. The valuation methodology allows for stochastic asset values and interest rates in a discrete time framework that can accommodate an arbitrary specification of the default event and boundary conditions. Numerical analysis based on binomial trees shows that these credit risk derivatives tend to be valued highest at middle maturities, conditional on a combination of both the time value of the credit risk debt and that of the derivative instrument on the credit risky debt is considered, decrease with the volatility interest rates and increase with the volatility of the firm.

Another credit risk derivative model developed in the context of models proposed by Merton (1974) [94] and Black and Cox (1976) [18] was postulated by Pierides (1997) [103]. In his paper Pierides (1997) [103] examines the structuring and valuation of credit risk derivatives that covers the losses of corporate bondholders from a widening in the spread above Treasuries at which they trade. In particular, this article considers derivatives structured as puts on the bond price or calls on the bond spread. A second feature of this paper is that it considers credit risk derivatives of the American type and postulates sev-

eral no early exercise scenarios. In addition, the default threshold for the coupon bonds is modelled endogenously coupled with the assumption of constant interest rates. The pricing properties of these options are derived with analytical methods and numerical analysis.

Kijima and Muromachi (2000) [82], in their paper, present a model for the valuation of a credit risk derivative whose payoff depends on the definition of the first to occur of a pre-specified list of credit events, particularly defaults. Their model postulates joint survival probability of occurrence times of credit events that is defined in terms of stochastic intensity processes under the assumption of conditional independence. Conditional on the default intensity following the extended Vasicek (1977) [118] short rate model they are able to derive closed form solutions for the valuation formulas of credit risk derivatives. The model framework has the flexibility to consider several extensions. For example, real world observable phenomena can be incorporated into the model such as the *recovery of market value* (RMV) assumption by Duffie and Singleton (1999) [42] or when a short rate model is used for the defaultable term structures the model can be calibrated to current market data.

## 1.4 Concluding Remarks

The aim of this dissertation is to give a precise and definitive account on the methodologies and techniques for the pricing of credit risk and credit risk derivatives. Although an exhaustive study on all the research published on this topic is beyond the scope of this work we, however, apportion purposeful attention to the classic and mature research. The thrust of the term *mature* emphasizes widely acclaimed research and increasingly efficient models that is representative of real observations. Nonetheless, we focus this exposition on structural models and reduced form models, the two conceptual parallel methodologies for the pricing of credit risk. In so far as credit risk derivatives, we review the mechanisms of three basic instruments and postulate two models to price credit risk derivatives with one model each set in the structural and reduced form framework, respectively.

Finally, a short note with respect to the dual use of terminology in this exposition. On several occasions we describe the identical concept or idea interchangeably with different terms. Some examples confirm. Risk-free is equivalent to zero credit risk. Default free is a synonym for risk-free or risk-less. Always in this work, risky explicitly refers to credit risk and not market risk. Bankruptcy is sometimes substituted to mean default.

In the context of this dissertation and by convention the concepts credit risk and default risk are often times used interchangeably although in literature a rigorous interpretation of default risk is deemed to be the risk that an obligor is unable to timeously honour payments of interest or principal on debt securities.

## Chapter 2

# Fundamentals of Credit Risk Modelling

The primary aim of this chapter is to introduce general contingent claim pricing concepts fundamental to the subjects treated throughout this dissertation. These are standard results in mathematical finance that can be found in Bingham and Kiesel (1998) [15] or Musiela and Rutkowski (1997) [97] and are intended to be a useful exposition on some of the main concepts and techniques in financial modelling that will be adapted to the pricing of credit risk and credit risk derivatives.

### 2.1 Basic Concepts and Assumptions

We consider a trading interval  $[0, T]$  for a fixed  $T > 0$  and a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  with a filtration  $(\mathcal{F}_t)_{0 \leq t \leq T}$ , generated by a  $d$ -dimensional Brownian motion, where  $\mathcal{F}_0$  is trivial,  $\mathcal{F}_T = \mathcal{F}$ , and which satisfies the usual conditions. We assume that there are a finite number of stochastic processes  $(S^0, S^1, \dots, S^k)$  where  $k \geq d$ , all of which are adapted semi-martingales defined by the stochastic differential equations (SDE) driven by the aforementioned Brownian motion.

We define a *market* as a set of traded assets denoted by a multi-dimensional price process  $S_t$ . The prices of these securities are non-negative and real-valued i.e.  $S_t \in (R^+)^d$ , where  $d$  is the number of assets in the market. The time  $t$  is taken to be a non-negative real-number with  $S_t = 0$  corresponding to some initial value.

We will assume that any time  $t > 0$  an agent participating in the market can *shortsell* assets. That is, he can sell assets without having owned them previously. The agent owns these assets and has the obligation to return them or pay them a later date.

A *portfolio* is a combination of assets. It is denoted by  $\phi \in R^d$ , which represents the total amount held of each asset. If a component  $\phi_k$  is negative, this indicates that the agent has short sold  $|\phi_k|$  of the  $k^{th}$  asset.

An asset  $(S_t)_j$  pays a *dividend* of  $\mathcal{D}_k$  at time  $t_k$  means that the owner of that asset at time  $t_k$  receives  $\mathcal{D}_k$  units of currency at time  $t_k$ .  $(S_t)_j$  pays *periodic dividends* with a yield  $q$  and frequency  $\Delta t$  means that the owner receives a dividend of  $q(S_t)_j \Delta t$  at every time  $i\Delta t$  for all integers  $i$ . If we take the limit where  $\Delta t$  approaches zero, we say that  $(S_t)_j$  pays *continuous dividends*. We say that dividends are *reinvested* when all of the cashflow that is received from dividends is used to buy more of the same asset at the trading market price. In the case of continuous dividends, an agent who purchases one security of  $(S_t)_j$  at time  $t_0$  will hold  $e^{q(t-t_0)}$  securities at any time  $t \geq t_0$ .

**Definition 1:** A *trading strategy* is a predictable vector process  $\phi_t = (\phi_t^0, \phi_t^1, \dots, \phi_t^k)$ . We also assume  $\mathbb{E}[\int_0^T \phi_t^2 \cdot d[S, S]_t] < \infty$ , although this assumption can be relaxed. Intuitively,  $\phi_t^i$  represents the amount of security  $i$  that we hold at time  $t$ .

**Definition 2:** The *value process*  $V_t(\phi)$  is defined as

$$V_t(\phi) = \sum_{i=0}^k \phi_t^i S_t^i$$

This is simply how much our total portfolio is worth at time  $t$ .

**Definition 3:** We say a *trading strategy* is *self-financing* if

$$V_t(\phi) = V_0(\phi) + \int_0^t \phi_u dS_u$$

A self-financing strategy requires that all wealth of our value process result only because of fluctuations from the price processes of the securities. In other words, there is no withdrawals of cash or injections of new funds from the value process after time 0. Throughout this dissertation, we will assume that all trading strategies are self-financed.

**Definition 4:** An *arbitrage strategy* is a trading strategy  $\phi$  with

$$\begin{aligned} V_0(\phi) &= 0 \\ \mathbb{P}(V_T(\phi) \geq 0) &= 1 \\ \mathbb{P}(V_T(\phi) > 0) &> 0 \end{aligned}$$

We say that the market allows an *arbitrage strategy* if there exists a self financing trading strategy  $\{\phi_t\}_{t \in [0, T]}$  such that its initial value is non-positive, and its value at the maturity date is non-negative and is positive with positive probability. Therefore, in our models, arbitrage opportunities should not exist and we need to specify conditions which eliminate arbitrage opportunities in our model.

**Definition 5:** We will define  $S^0$  to be the *savings account*, that is

$$dS_t^0 = r_t S_t^0 dt$$



with  $S_0^0 = 1$  where  $r_t$  is an optional process and  $\int_0^t r_t dt < \infty$  a.s. Thus a *savings account* is an asset which continuously earns interest at a spot rate of  $r_t$ . In general,  $r_t$  can be stochastic. We assume that this value will always be paid at time  $t$  if the owner chooses to sell, implying that there is no risk of default. We will assume throughout this dissertation that there exists a non defaultable savings account in the market.

**Definition 6:** The *discounted price process* for asset  $i$  is  $\tilde{S}^i := \frac{S^i}{S^0}$ . The asset which is used for discounting (in this case  $S^0$ ) is known as the *numéraire asset*.

We make the following assumptions throughout this dissertation, unless we explicitly state alternate assumptions.

1. All the assets can be bought and sold at any time in any quantity.
2. All assets can be traded without paying *transaction costs*. A transaction cost is an additional fee which is charged when one is buying or selling an asset. This can be represented by requiring that an agent buy the asset at a premium and sell it at a discount. The existence of this buy-sell action is usually referred to as a *bid-ask spread*.
3. All transactions take place instantly, and the payments are received at the time the transactions occur.
4. All promised cash-flows are received with absolute certainty without any risk of default.
5. All agents behave rationally in the sense that they prefer more wealth to less and that they will preferably pursue trading strategies which would maximize their expected future wealth.
6. All agents are aware of all the available information about the market at any given point in time.
7. There are no arbitrage opportunities.

These assumptions are standard in literature. The final assumption, that there are no arbitrage opportunities, forms the foundation of asset pricing theory. This assumption generally holds in real markets on a macroscopic scale for the following reason. If there was an arbitrage opportunity then most agents would take as large a position as possible in whatever portfolio gave a positive probability of profit without any initial cost or risk of loss. This increased demand would drive the price of that position upwards until the arbitrage opportunity was eliminated.

The following theorem gives a sufficient condition for the absence of arbitrage opportunities.

**Theorem 1:** There exists a measure  $\mathbb{Q}$  equivalent to  $\mathbb{P}$  such that the discounted price process  $\tilde{S}^1, \tilde{S}^2, \dots, \tilde{S}^k$  are  $\mathbb{Q}$  martingales if and only if no-arbitrage strategies exist.

For a proof, see Harrison and Pliska (1981) [62]. We will refer to  $\mathbb{Q}$  as a *martingale measure* for the numéraire asset  $S^0$ . In the above, we can actually use any of the other assets as a numéraire provided the numéraire asset remains positive with probability 1. This sometimes leads to simpler valuation formulae. For instance, when valuing options on bonds, it is often convenient to use another bond as a numéraire asset. However, the most common numéraire is the savings account.

If we let  $\tilde{S}_t = \frac{S_t}{S_t^0}$  to be the discounted price process and  $\tilde{V}_t(\phi) = \frac{V_t(\phi)}{S_t^0}$  be the discounted value process, it follows from Itô's formula that a trading strategy is self-financing if and only if

$$d\tilde{V}_t(\phi) = \sum_{i=1}^k \phi_t^i d\tilde{S}_t^i$$

We notice that  $\tilde{V}_t$  is a martingale under the martingale measure as long as  $\phi$  satisfies some bounded process.

Of particular interest in financial markets is how an agent can accurately price and hedge contingent claims. A *contingent claim* is an asset with the following properties:

1. It gives the owner the right to claim a predetermined cashflow called a *payoff*, at some future date  $T$  called an *expiration date*, or at some set of dates  $\{t_i\}$  with  $\sup(\{t_i\}) = T^1$
2. The payout is a function of the value of other assets at the time which the cashflow is claimed. It is claimed at time  $\tau$  then the owner receives a cashflow of  $X(S_\tau, \tau)$  at time  $\tau$ .

Contingent claims are often referred to as *options*, *derivative securities*, or *financial derivatives*. For our purposes, a *contingent claim* is simply a  $\mathcal{F}_T$ -measurable random variable  $X$ . We are interested in whether there exists a self-financing trading strategy  $\phi$  which replicates  $X$  i.e.  $V_\phi = X$ ,  $\mathbb{P}$ -a.s. We will call this strategy a *replicating strategy* for  $X$ . A *replicating strategy*  $\phi_t$  is a trading strategy on the traded assets which has value  $V_t$ , for  $t$  at or before the claims expiration date. That is

$$V_t = \phi_t S_t$$

for all values of  $t$ . Pursuing a replicating strategy is referred to as *hedging*. A replicating strategy is often referred to as a *hedging strategy*. If  $\phi_t$  is a hedging strategy then the value of one of the components  $(\phi_t)_k$  at time  $t$  is called the *hedge ratio* of the contingent

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<sup>1</sup>In general,  $T$  can be a bounded stopping time as well as a fixed time. If the cashflow can only be claimed at a fixed expiration date then the contingent claim is called a European contingent claim. If the cashflow can be claimed anytime before a expiration date then it is called an American contingent claim.

claim with respect to the asset  $(S_t)_k$  at time  $t$ .

**Theorem 2:** Assume a martingale measure exists. If  $\mathbb{Q}$  is extremal in the set of martingale measures, then for every  $X \in \mathcal{L}^1([0, T], \mathbb{Q})$ , there exists a replicating strategy for  $X$ .

This is just a restatement of the martingale representation theorem. In other words, if a unique martingale measure exists, then every integrable contingent claim can be replicated. In the case where replication strategies exist, the price of the contingent claim,  $\beta_t(X)$ , at  $t < T$  must be the same as the replicating strategy at  $t$ . Similar to a discounted price process, the price of a discounted contingent claim at  $t$  must be the same as the value of a discounted value process at  $t$ . We can write

$$\frac{\beta_t(X)}{S_t^o} = \tilde{V}_t(\phi) = \mathbb{E}_{\mathbb{Q}}[\tilde{V}_T(\phi)|\mathcal{F}_t] = \mathbb{E}_{\mathbb{Q}}\left[\frac{X}{S_t^o}\right]$$

In some of the examples we may work with replicating strategies that may not always exist. This is not necessarily a problem since our main concern is to price a claim rather than replicating or hedging them. We want to ensure that the price we quote on a contingent claim will not result in arbitrage opportunities.

**Theorem 3:** Let  $\mathbb{Q}$  be a martingale measure. Then an arbitrage-free price for  $X$  at time  $t < T$  is given by

$$\beta_t(X) = S_t^0 \mathbb{E}_{\mathbb{Q}}\left[\frac{1}{S_T^0} X | \mathcal{F}_t\right]$$

We notice that under the equivalent martingale measure  $\mathbb{Q}$ , the arbitrage-free prices of contingent claims satisfy the following stochastic differential equation:

$$d\beta_t(X) = r(t)\beta_t(X)dt + dB_t$$

where  $B$  is a  $\mathbb{Q}$ -martingale.

The market generally admits several martingale measures and if  $\beta_t(X)$  is an arbitrage free price for  $X$  at time  $t$ , it may not be unique if several martingale measures exist. Our main concern is to model prices accurately and we should choose a measure that replicates prices observed in the market place. Then for the purposes of this dissertation we will assume that a unique martingale measure exists.

There are no restrictions on Theorems 1, 2 and 3. That is, they should hold in very general probability spaces. This means that the security prices can follow any dynamics, including jump processes, and with the appropriate adjustment of the filtration, the fundamental relationship between the existence of an equivalent martingale measure and the exclusion of arbitrage still holds. The fact that these theorems still hold in the case of jumps is crucial to our analysis in the following chapters.

## 2.2 Interest Rate Modelling and Bond Markets

The primary focus of this dissertation is the credit risk market for bonds and other fixed income products which depend primarily on interest rates. We consider a trading horizon  $[0, T^*]$  to follow conventional notation.

**Definition 7:** A *zero coupon bond* with maturity  $T \leq T^*$  is a contingent claim which pays one unit at time  $T$ , then accrues at the instantaneous rate  $r_u$  for  $u \in [T, T^*]$ . A zero coupon bond pays  $\exp(\int_T^{T^*} r_u du)$  at time  $T^*$ . The price of a zero coupon bond with maturity  $T \leq T^*$  at time  $t \leq T$  is denoted by  $P(t, T)$ .

In general we assume that zero coupon bonds with maturity  $T$  exist for all  $T \in [0, T^*]$ , and for each fixed maturity  $T$ , the price process  $(P(t, T) : 0 \leq t \leq T)$  is optional with  $P(t, t) = 1$  for all  $t$ .

We assume (in this chapter) that the payment will be made with absolute certainty. In general, if the payment is not made then the issuer of the bond is said to *default*.

A general bond may have *coupons*. These are payments of some amount  $c_i$  which are paid at times  $t_i$ , where the  $t_i$ 's are less than equal to the final maturity date  $T$ . The bond will also pay some *principal* amount at maturity.<sup>2</sup>

When working with zero coupon bonds, it is not convenient to specify the dynamics of bond prices. It is preferable to work with interest rates, and to derive the dynamics of the bond prices from the interest rates.

**Definition 8:** The *short rate* or *spot rate*  $r_t$  is the instantaneous rate of interest that applies at time  $t$ , contracted at time  $t$ .

The risk-less asset defined in the first section assumes continuous compounding at the short rate  $r_t$ . We now denote this risk-less asset as the savings account process  $B(t) = \exp(\int_0^t r_u du)$ . By specifying the dynamics of  $r_t$  allows us to price zero coupon bonds using Theorem 3.

The process  $r_t$  is generally taken to be a diffusion process defined by the stochastic differential equation:

$$dr_t = \mu(r_t, t)dt + \sigma(r_t, t)dB_t$$

where  $B_t$  is a one-dimensional Brownian motion under a fixed martingale measure  $\mathbb{Q}$ . Then by Theorem 3, the price of a zero coupon bond at time  $t$  is given by

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<sup>2</sup>The principal amount is generally referred to as the *par value* of the bond. This is usually taken to be 100.

$$\begin{aligned}
P(r_t, t) &= \mathbb{E}_Q \left[ \exp \left( - \int_t^{T^*} r_u du \right) \exp \left( \int_T^{T^*} r_u du \right) \middle| \mathcal{F}_t \right] \\
P(r_t, t) &= \mathbb{E}_Q \left[ \exp \left( - \int_t^T r_u du \right) \middle| \mathcal{F}_t \right]
\end{aligned}$$

It has been shown that, in this formulation, the value of a zero coupon bond  $P(t, T)$  at time  $t$  maturing at time  $T$  must be the solution to the partial differential equation:

$$\frac{\partial P}{\partial t}(r, t) + \frac{1}{2} \sigma^2(r, t) \frac{\partial^2 P}{\partial r^2}(r, t) + \left( \nu(r, t) + \lambda(r, t) \sigma(r, t) \right) \frac{\partial P}{\partial r}(r, t) - rP(r, t) = 0$$

$$P(r, T) = 1$$

according to the *Feynman-Kac* formula and where  $\mu = \nu + \lambda \sigma$  is a risk-neutral drift and  $P(r, T)$  is the boundary condition. Where applicable, we will model  $\mu$  directly and assume that  $r$  is defined under the risk-neutral measure.

The short rate is not the only interest rate available in the market, that is, the interest rate is different for different maturities. This implies that an effective model of interest rates should be able to include all the information about the different rates for different maturities. In order to do this we will focus on the forward rates, which will allow initial bond prices to be inputs.

**Definition 9:** The *instantaneous forward rate*  $f(t, T)$  is defined as the instantaneous interest rate that applies at time  $T$  contracted at the current time  $t \leq T$ . We refer to  $T$  as the time of maturity. If the bond prices are sufficiently smooth, we can define the forward rate as

$$f(t, T) = - \frac{\partial \ln P(t, T)}{\partial T} \quad (2.1)$$

Intuitively, we can interpret  $f(t, T)$  as the interest rate applying over the infinitesimal time interval  $[T, T + dT]$  which can be locked in at time  $t$ . Because at time  $t$ , there is a continuum of forward rates as a *forward curve*,  $f(t, \cdot) : [t, T^*] \rightarrow R$ .

We can relate bond prices to forward rates through the following formula:

$$P(t, T) = \exp \left( - \int_t^T f(t, u) du \right) \quad (2.2)$$

We notice that at time 0, we can define the forward rates  $f(0, T)$  using (2.1) and we have initial bond prices consistent with the market prices. Heuristically, (2.1) must hold for there to be no arbitrage. In this dissertation we will always be working in a framework which emphasizes the dynamics of forward rates, rather than the short rate.

## 2.3 Hazard Rate Modelling and Intensity Processes

Fundamental to the pricing of credit risk in reduced form models is the characterization of a point event which we frequently refer to as the default event and that which occurs after an arbitrary length of time. More precisely, we define this arbitrary length of time to be a random variable called the default time,  $\tau$ . This random variable forms the basic construct for the valuation of financial instruments subject to credit risk. In addition, we show the hazard rate function and counting processes as the building blocks for reduced-form intensity models.

**Definition 10:** There exists on the filtered probability space  $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$  a jump process  $N(t)$  with deterministic intensity  $\lambda(t)$ . Default occurs at the first jump time of  $N(t)$ , formally

$$\tau := \inf\{t \geq 0 \mid N(t) = 1\}$$

The probability of no default occurring between time  $t$  and  $T$  is given by

$$\mathbb{P}(N(T) - N(t) = 0) = e^{-\int_t^T \lambda(u) du}$$

Let  $\tau$  be a stopping time and  $F(T) := \mathbb{P}(\tau \leq T)$  be its distribution function. This implies the distribution of the default time is given by

$$\mathbb{P}(\tau \leq T) = 1 - e^{-\int_0^T \lambda(u) du}$$

and its density is expressed as:

$$f(t) = \lambda(t) e^{-\int_0^T \lambda(u) du}$$

The distribution  $F(T)$  is just one method to specify the distribution of the default time. A related and more commonly used method to characterize the default arrival risk is the hazard rate function which gives the instantaneous default probability at time  $t$  and is defined as:

$$h(t, T) := \frac{f(t, T)}{1 - F(t, T)} = \lambda_t \quad (2.3)$$

where  $F(T) := \mathbb{P}(\tau \leq T | \mathcal{F}_t)$  is the conditional distribution of  $\tau$  with respect to the reference filtration  $\mathbb{F}$ , and  $f(t, T)$  is the corresponding density.

The hazard rate function is alternately known as the intensity which can be expressed in terms of (2.3) as the conditional default arrival rate, given no default:

$$\lim_{s \rightarrow 0} \frac{\mathbb{P}(\tau \in (t, t+s] \mid \tau > t)}{s} = \frac{f(t)}{1 - F(t)} = \lambda_t$$

The Poisson process is a familiar example of a counting process that has an intensity  $\lambda$  where  $\lambda$  is a predictable non-negative process that satisfies  $\int_0^t \lambda(s) ds < \infty$  a.s. for all  $t$ . As such the Poisson process has been adopted as a practical approach of modelling

default risk in the reduced form model framework.

In an intensity based model the time of default of an entity is modelled as the first jump of the designated counting process, for example, a Poisson process with intensity  $\lambda > 0$ . In credit risk models the default intensity is parameterized from observable variables such as volatility measures, exchange rates and bond yield spreads. We are now ready to give a formal definition of a Poisson process.

**Definition 11:** After drawing a sequence  $(Y_i)$  of independent exponential random variables of parameter 1, we let  $\tau_n$  be the partial sum of the first  $n$  terms of the sequence

$$\tau_n = \sum_{i=1}^n Y_i$$

and define a counting process associated with that sequence as a stochastic process  $N_t$  given by:

$$N_t = \sum_{n=1}^{\infty} I_{\{\tau_n \leq t\}}$$

This process is a standard Poisson process of parameter 1.

This leads us to the definition of a homogeneous Poisson process.

**Definition 12:** The counting process  $N_t, t \geq 0$  is said to be a homogeneous Poisson process having parameter  $\lambda$ ,  $\lambda > 0$ , if:

1.  $N_0 = 0$
2.  $N_t, t \geq 0$  has independent increments
3. the number of events occurring in any interval of length  $\Delta t$  is Poisson distributed with mean  $\lambda \Delta t$ . That is, for all  $t, \Delta t \geq 0$

$$\mathbb{P}\{N_{t+\Delta t} - N_{\Delta t} = n\} = e^{-\lambda \Delta t} \frac{(\lambda \Delta t)^n}{n!} \quad n = 0, 1, 2, \dots$$

We can generalize the homogeneous Poisson process by specifying the default intensity,  $\lambda(t) > 0$ , to be time dependent in which case we can now define an inhomogeneous Poisson process.

**Definition 13:** The counting process  $N_t, t \geq 0$  is said to be a non-stationary or non-homogeneous Poisson process with intensity function  $\lambda(t), t \geq 0$  if:

1.  $N_0 = 0$
2.  $N_t$  has independent increments

3.  $\mathbb{P}\{N_{t+s} - N_t \geq 2\} = o(s)$
4.  $\mathbb{P}\{N_{t+s} - N_t \geq 1\} = \lambda(t)s + o(s)$

That is for all  $s, t \geq 0$ , a non-homogeneous Poisson process  $N$  with  $\lambda(\cdot)$  satisfies

$$\mathbb{P}\{N_{t+s} - N_s = n\} = \frac{\left(\int_s^t \lambda(u) du\right)^n}{n!} e^{-\int_s^t \lambda(u) du} \quad n = 0, 1, \dots$$

Finally, we let the intensity  $\lambda$  depend on a stochastic process and therefore can also be stochastic and we can thus obtain the so-called Cox process.

A Cox process  $U$  with intensity  $\lambda = (\lambda_t)_{t \geq 0}$  is a generalization of the non-homogeneous Poisson process in which the intensity is allowed to be stochastic with the caveat that conditional on the realization of  $\lambda$ ,  $U$  is a non-homogeneous Poisson process. With this restriction  $U$  is also called a conditional Poisson process or a doubly stochastic Poisson process.

To give an example, we can assume  $\lambda = (\lambda_t)_{t \geq 0}$  follows a diffusion process of the form

$$d\lambda_t = \mu(\lambda_t, t)dt + \sigma(\lambda_t, t)dB_t$$

where  $B$  is the Brownian motion. Another example is to assume that the intensity is a function of a set of state variables (exchange rates, interest rates, bond yields, etc.)  $Y$ , i.e.  $\lambda_t = \lambda(Y_t, t)$ .

## 2.4 The Black-Scholes Model

Thus far we have tried to present the basic concepts and the main intuitive ideas associated with replication and arbitrage strategies. These concepts and ideas, when developed further, give rise to the general theory of option pricing. The standard theory has much to do with technical conditions and if these conditions are surreptitiously added when constructing an arbitrary well-purposed model then important intuitive ideas can be omitted. On the other hand, if we choose to introduce the technical conditions sparingly, or defer them altogether, we may leave the reader aloof as to the nature and extent of the technical theory that is applicable to a particular model. Nonetheless, throughout this dissertation we will present an exposition with intuitive ideas supported by sufficient technical detail. As a case in point we present the Black-Scholes model.

In this section we consider the model by Black and Scholes (1973) [20] in their seminal work on option pricing. The original Black-Scholes model presented a partial differential equation approach to pricing and hedging contingent claims on securities which pay no dividends, continuous dividends and foreign currencies. We state this equation as an example of how to price and hedge contingent claims in a complete market. We will not always use this equation with its original representation of state variables directly in



this dissertation but instead variants of it. Nonetheless, it is instructive to note that the techniques and methods we demonstrate will be used frequently throughout the dissertation. We will also include it as a symbolic reference as it was the first method to price and hedge contingent claims using a dynamically adjusted replication strategy and was a starting point for most subsequent work.

The initial step to pricing derivatives is to define the underlying economy for the model setup. In general, an economy is defined by two components. The first, is a model for the generation of the state prices of the assets of the economy. The second, is to define a set of trading strategies that is admissible in the economy. Throughout this section we define a finite trading interval to be  $\tau = [0, T]$ .

Let the economy,  $\epsilon$ , be defined by a set of primary securities and a set of self-financing trading strategies. We assume that the economy consists of two securities, with one asset  $S_t$  and a money market account  $B_t$ . The money market account is defined by the ordinary differential equation:

$$dB_t = rB_t dt \quad \text{with} \quad B_0 = 1$$

$r$  is the risk-less interest rate and is assumed constant over  $[0, T]$ . The risk neutral dynamics of the stock price process is assumed to follow a geometric Brownian motion and is given as:

$$\begin{aligned} dS_t &= rS_t dt + \sigma S_t dB_t \\ S_0 &= s \end{aligned} \tag{2.4}$$

The drift  $r$  and the diffusion  $\sigma$  are assumed constant on  $\tau$ . We choose  $s$  as an arbitrary starting value.  $B_t$  is a standard Brownian motion defined on the filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{Q})$ .

Now, let us consider a European contingent claim with maturity  $T$  and payoff function  $F(S_T)$ . If at time  $t$  the price of the asset is  $S_t$  then at time  $t$  we can represent the value of this claim as  $V(S_t, t)$ . We have that  $V$  must be the solution to the Black-Scholes equation:

$$\begin{aligned} V_t + \frac{1}{2}\sigma^2 S^2 V_{ss} + rSV_s - rV &= 0 \\ V(S, t = T) &= F(S) \end{aligned} \tag{2.5}$$

This equation is derived from constructing a continuous time risk-less hedge portfolio. The hedge ratios are given by  $V_s(S_t, t)$  and equivalently this quantity is referred to as  $\Delta(S_t, t)$ . This replication strategy is often called a *delta hedge*.

Notice that the solution to the Black-Scholes partial differential equation (2.5) is a function of  $S_t$  and so to is  $S_t$  a solution to the SDE given by (2.4). A convenient approach

to solve the Black-Scholes PDE is to make use of the *Feynman-Kac* representation for parabolic differential equations. If we apply the Feynman-Kac formula to (2.5) then we have

$$V(S, t) = \mathbb{E}_{S,t}^Q \left[ e^{-\int_t^T r_u du} F(S_T) | \mathcal{F}_t \right]$$

Note that the value of  $V$  is the expectation of  $S_t$  under the measure,  $\mathbb{Q}$ . This measure  $\mathbb{Q}$  is called the risk-neutral martingale measure and it gives the average return of  $S_t$  when the money market account is reinvested continuously. The fact the economy is complete implies that  $\mathbb{Q}$  is the unique equivalent martingale measure. We are now ready to state the following proposition taken from Ammann (2001) [4].

**Proposition:** The economy,  $\epsilon$  admits a martingale measure  $\mathbb{Q}$  which is called the risk-neutral measure. Under  $\mathbb{Q}$  the solution to the SDE in expression (2.4) is given by:

$$S_t = S_0 e^{(r - \frac{1}{2}\sigma^2)t + \sigma B_t} \quad (2.6)$$

**Proof:** see Ammann (2001) [4] P27.

We can now say that, in general, under the risk-neutral measure the expected returns of all traded assets are the same as that for a riskfree money market account. For the purposes of this dissertation we will assume that the economy is complete then the unique equivalent martingale measure will be the same as the risk-neutral measure.

Thus far in the analysis of the Black-Scholes (1973) [20] model we have considered the value function of a contingent claim whose payoff is an arbitrary function of a stock price process. In particular, we specified that the stock price process to follow a standard geometric Brownian motion and have shown this value function satisfies the Black-Scholes (1973) [20] equation. Next, as a specific example of a contingent claim we will state the value function of a standard put option which is given by the Black-Scholes (1973) [20] formula.

A European contingent claim with payoff  $V_T = (K - S_T)^+$  written on a stock  $S$  with maturity date  $T$  and strike  $K$  is called a *put option*. Under the risk-neutral dynamics its price is given by:

$$V_t = K e^{-r(T-t)} N(-d_2) - S_t N(-d_1) \quad (2.7)$$

where

$$d_1 = \frac{\ln \frac{S_t}{K} + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\ln \frac{S_t}{K} + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$N$  is the cumulative normal distribution function.

In the work that follows, our purpose is now to extend the modelling building blocks presented in this chapter to the pricing of credit risk and credit risk derivatives.

University of Cape Town

## Chapter 3

# On Structural Models of Pricing Credit Risky Bonds

### 3.1 Introduction

In this chapter we study structural models of credit risk pricing in which the corporate obligor's ability to meet its debt obligation is explicitly modelled. A key assumption of the model is that credit risk arises from the likelihood of default. Generally, we say that the corporate is in default if the assets of the firm are less than the outstanding debt where debt is signalled by some pre-specified barrier.

The pricing of credit risky bonds in a continuous time setting has been widely researched since the seminal work by Merton (1974) [94]. More precisely, Merton (1974) [94] extended the Black-Scholes (1973) [20] option pricing methodology to credit risk in a financial economics setting so as to value the component parts of a firm's balance sheet structure. This framework is widely known as the *structural model* or *firm value* approach since it restricts the capital structure of the firm to the two main claimants, equity owners and bondholders, to the value of the assets of the firm. The structural model approach makes explicit assumptions about the dynamics of the firm's asset value, its bondholders and equity owners as well as its capital structure. In addition, this seminal research has brought to the fore the fundamental contribution that components of a firm's capital structure can be modelled as contingent claims on the value of the firm's assets.

On analysis of the classic Merton (1974) [94] firm value model we observe that the default event can only be identified at maturity of the debt obligation. Clearly, this is a model specific assumption as in practice bondholders have right to exercise bond indenture provisions such as debt covenants if the value of the firm breaches some pre-specified level. Moreover, empirical regularities show that credit spreads generated by the Merton (1974) [94] model appear to be too low. Consequently, Black and Cox (1976) [18] tried to overcome this limitation by calibrating the model to default as a conditional instantaneous event. This bears the plausible economic reality that the firm continues to operate

as long as it remains solvent. As such this modification by Black and Cox (1976) [18] has shown a new class of structural models termed first passage time models where a firm can default during the tenure of the bond obligation. Nonetheless, spreads forecast by the Black and Cox (1976) [18] model are higher than the Merton (1974) [94] model but still under-perform relative to market observations.

A review of the literature shows some stylized facts on structural models. The first widely acclaimed first passage time model was introduced by Black and Cox (1976) [18] and their work added several new features to the valuation of corporate debt. In particular, they presented a theoretical analysis of bond indenture provisions that analyzed the effect of safety covenants on the value and nature of financial instruments. The value of the firm is allowed to vary with time and if the pre-specified default threshold is breached then the safety covenants allow the bondholders the right to reorganize or foreclose on the firm. In addition, their work has shown the basis for a stochastic model of bankruptcy even though they did not include bankruptcy costs in their model. Their research has shown that a safety covenant such as bankruptcy costs can potentially reduce the price of credit risk in contingent claim valuation by a considerable amount if it is associated with a judiciously chosen default boundary.

Subsequent work to the Black and Cox (1976) [18] model has served to expound on the different features of the first passage time model. For example, Briys and de Varenne (1997) [23] in their analysis of previous structural models alluded to two apparent anomalies in some of the models. First, in the model proposed by Nielsen, Saá-Requejo and Santa-Clara (1993) [99] they identify an anomaly that at default, either prematurely or at maturity, the payoff to the bondholders, in some cases, can be greater than the value of the firm. This occurs since the payoff at default is independent of the stochastic default threshold and the value of the firm. The second anomaly arises, as for example, in the Longstaff and Schwartz (1995) [91] model when the corporate bond reaches maturity. In this model the firm can find itself in a solvent state relative to the default threshold at maturity but at the same time have insufficient value of the firm to redeem the bond issue. They correct these anomalies by suggesting that the default threshold and recovery rate at default be defined judiciously so that the payoff at default is always less than the value of the firm.

In this chapter we study the pricing of credit risk in the firm value framework. We aim to show a first passage time model as first developed by Schönbucher (2000) [112] and this model incorporates several of the desirable features mentioned in literature. For example, some distinguishing features of the model include bankruptcy costs at reorganization of the firm, stochastic interest rates that is correlated with the firm value and a judiciously chosen default threshold and recovery value so that in the event of default the payoff to the bondholders does not exceed the value of the firm. In addition, we show two variants of this first passage time model (i) a constant default boundary model and (ii) a deterministic default boundary model, with constant interest rates. Moreover, we construct the model framework using simple and specific assumptions to price credit risky bonds. Con-

sequently, we are able to derive closed form solutions for all models presented in this study.

The remainder of this chapter is organized as follows. In section 3.2 we set out a concise overview of the classic Merton (1974) [94] firm value model. Section 3.3 presents the model setup. First, in section 3.4, we show a stochastic default boundary model. Second, in section 3.6, we show a constant default boundary model. Third, in section 3.7, we show a deterministic default boundary model. In section 3.8 we implement each model so as to simulate and analyze credit spreads. In section 3.9 we implement each model so as to simulate and analyze the probability of default. In section 3.10 we describe a strategic analysis of structural models. Section 3.11 concludes.

## 3.2 Merton Firm Value Model

The Merton (1974) [94] methodology relies on several assumptions many of which are derived from the Black-Scholes (1973) [20] option pricing model and as such allows for consistency in the application of option pricing techniques to the modelling of credit risk. In particular, the value process of the assets of the firm follow a geometric Brownian motion and is given as:

$$dV_t = \mu V_t dt + \sigma V_t dB_t$$

with  $\mu$  being the drift of the process,  $\sigma$  the diffusion coefficient of the underlying asset and  $B_t$  a standard Brownian motion. The diffusion coefficient is assumed to have the additional characteristic of being constant over time. The capital structure of the firm is defined by two types of claims: risky debt,  $D$ , and equity,  $S$ . The debt is represented by a single non-callable zero-coupon bond of par value  $F$  and is due at maturity  $T$ . Figure 3.1 shows an illustration of the evolution of the asset value over the time horizon. The dashed horizontal line shows the par value of the liability and is alternately described as the default threshold. Clearly, if maturity is at the horizon time  $T$  then in the Merton (1974) [94] model the firm is in default of its debt obligation.

When the bond issue matures at time  $T$ , the bondholders will receive the par value  $F$  that is owing to them with the caveat that the value of the firm's assets is sufficient to honor this debt i.e.  $V_T > F$ , the equity owners then receive the balance of the assets,  $V_T - F$ . On the other hand, if the debt covenants allow for the absolute priority rule and the firm's assets at maturity is not sufficient to meet the bondholders claims i.e.  $V_T < F$ , the bondholders can immediately claim all the assets of the firm leaving the equity owners with no claim. In the event that  $V_T < F$  we say the firm has defaulted on its debt and in this case the bondholders can take over the firm. Accordingly, we can summarize the payoffs for the different contingent claims under the binary states of default and no default at maturity as set out in Table 3.1.

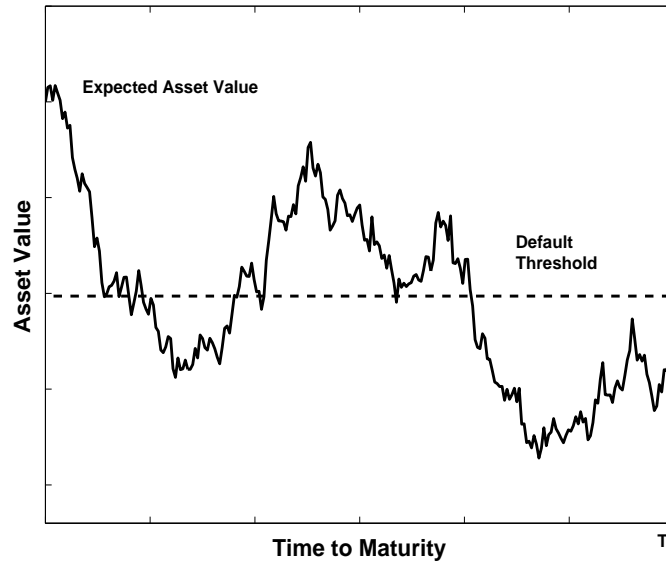


Figure 3.1: Expected Asset Value and Default Threshold at Horizon

On analysis of Merton's (1974) [94] model we observe that each payoff function listed in Table 3.1 can be written as the payoff on a European contingent claim and as such we can employ the option theoretic methodology to price credit-risky debt. In particular, the time  $T$  value of the firm's equity can be expressed as:

$$S_T = \max(0, V_T - F) \quad (3.1)$$

Similarly, at maturity,  $T$ , the payoff received by the bondholders can be expressed as:

$$D_T = \min(V_T, F) = F - \max(F - V_T, 0) \quad (3.2)$$

Moreover, each term in (3.2) underscores the conjunction between credit risky debt and contingent claim analysis. The underlying economic interpretation of (3.2) can be viewed as credit risky debt is equivalent to a portfolio with (i) a long position in a default-risk free zero coupon bond with par value  $F$  and (ii) a short position in a European put option on the assets of the firm with strike  $F$ .

The payoff functions in expressions (3.1) and (3.2) provides the key insight to value equity and debt as contingent claims on a firm's assets. The time  $t$  value of equity is derived as a European contingent claim and stated as the Black-Scholes (1973) [20] call option pricing formula:

$$\begin{aligned} S_t &= BS_c(\sigma, T - t, F, r, V_t) \\ &= V_t N(d_1) - F e^{-r(T-t)} N(d_2) \end{aligned} \quad (3.3)$$

	<i>ASSETS</i>	<i>DEBT</i>	<i>EQUITY</i>
Default	$V_T < F$	$V_T$	0
No Default	$V_T \geq F$	$F$	$V_T - F$

Table 3.1: Contingent Claim Payoffs on the Firm's Debt at Maturity

where  $N(\cdot)$  is the cumulative normal distribution function and

$$\begin{aligned}
 d_1 &= \frac{\ln \frac{V_t}{F} + (r + \frac{1}{2}\sigma_v^2)(T - t)}{\sigma_v \sqrt{T - t}} \\
 d_2 &= d_1 - \sigma_v \sqrt{T - t}
 \end{aligned} \tag{3.4}$$

We know that the time  $t$  value of risky debt is equivalent to the value of a risk-less bond less the value of the credit risk put option. Similarly, we can apply the Black-Scholes (1973) [20] pricing methodology to derive the time  $t$  value of the risky debt as:

$$\begin{aligned}
 D_t &= Fe^{-r(T-t)} - BS_p(\sigma, T - t, F, r, V_t) \\
 &= Fe^{-r(T-t)}N(d_2) - V_tN(-d_1)
 \end{aligned} \tag{3.5}$$

and the value of the Black-Scholes credit risk put option is derived to be:

$$BS_p = Fe^{-r(T-t)}N(-d_2) - V_tN(-d_1)$$

with  $d_1$  and  $d_2$  defined as in (3.4). Clearly, the value of the credit risk put option completely represents the price differential between risk-free and credit risky debt. In addition, the credit spread between risky and risk-free debt is also associated with the value of the credit risk put option,  $BS_p$ . The underlying determinants of the value of the credit put option is the risk-free interest rate of the firm value and the volatility processes, respectively. Both these state variables are deemed to be constant in this setting. We observe that if the risk-free interest rate decreases the credit spread between risky debt and risk-free debt must increase, since lower risk-free rates makes the credit put option more expensive. Similarly, if the volatility of the value of the firm decreases, the spread between risky debt and risk-free debt correspondingly decreases and in this scenario the value of the credit risk put option is cheaper.



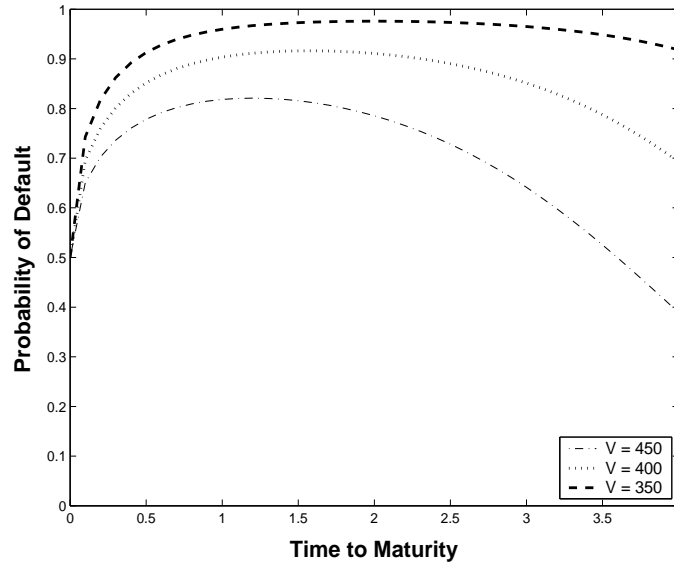


Figure 3.2: Term Structure of Risk-Neutral Probability of Default in Merton Model, varying Firm Value  $V$

The term structure of the firm value process in Merton's (1974) [94] model provides an appropriate method of expressing the arrival of default risk. The distribution of the probability of default is implicitly modelled in Merton's (1974) [94] model by assuming that the probability of default increasingly converges to an arbitrary but predictable stopping time,  $\tau = T$ . Intuitively, this means that the cumulative default probabilities converge to the value one as the value of the assets of the firm approaches the value of the default threshold.

From our description of the asset dynamics we can derive time  $t$  explicit formula for the unconditional probability of default under the risk-neutral probability measure  $\mathbb{Q}$  as:

$$\begin{aligned}
 \mathbb{Q}[\tau = T] &= \mathbb{Q}[V_T < F] \\
 &= \mathbb{Q}\left[\tilde{B}_T < \frac{\ln \frac{F}{V_t} + (\frac{1}{2}\sigma_v^2 - r)(T - t)}{\sigma_v \sqrt{T - t}}\right] \\
 &= 1 - N(d_2)
 \end{aligned}$$

where  $B_T$  is  $N(0, T)$  and  $d_2$  is as given previously. Figure 3.2 illustrates the typical term structure for the probability of default at varying values of the firm.

In this simplest structural model Merton (1974) [94] applies the economic argument that a firm defaults when the asset value of the firm,  $V_T$ , falls below the value of the debt obligation. It follows that the risk-neutral probability of default is just the probability of

an event that measures the expected value of a firm's assets at time  $T$  less the value of the bond obligation divided by the volatility of the firm's asset value. This conceptual argument is termed the *distance to default* and describes the number of standard deviations by which the assets exceeds its liabilities.

Suppose we have two identical bonds that have the same tenure, coupon rate and issued at the same time  $t$  but with one being default-free and the other being default risky. At any arbitrary time  $t$  the value of the risk-free bond is greater than the value of the credit risky bond. This differential is often termed *credit yield spread* or *credit spread*. The credit spread can be interpreted as the additional yield over the risk-free yield demanded by the bond investors for assuming the potential losses due to default of the bond obligor. The Merton (1974) [94] model shows the spread between the yield on the risky bond and the risk-less interest rate as:

$$s(t, T) = -\frac{1}{T-t} \ln \left[ N(d_2) + \frac{1}{d} N(-d_1) \right]$$

where  $d = \frac{F}{V_t} e^{-r(T-t)}$  is a measure of leverage,  $d_1$  and  $d_2$  are as given previously. Figure 3.3 illustrates the typical term structure of credit spreads at varying values of firm leverage.

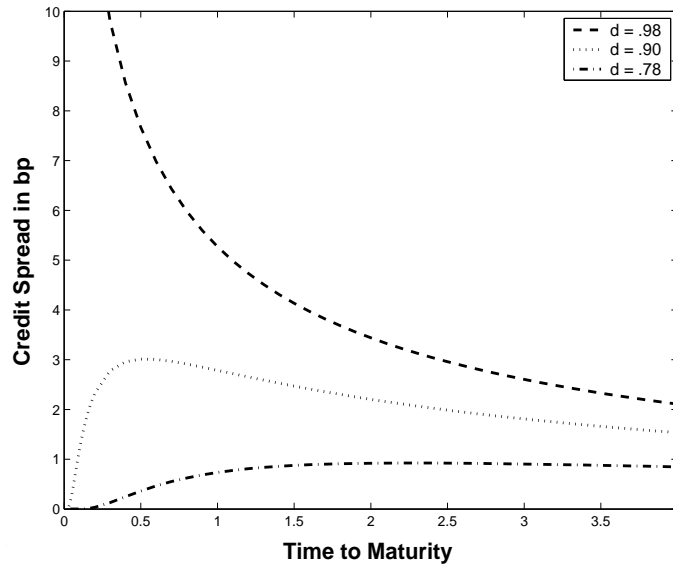


Figure 3.3: Term Structure of Credit Spreads in Merton Model with varying Firm Leverage  $d < 1$

A crucial realization of this model is that default is a predictable event. Then if the credit spreads evolve according to the underlying model dynamics they would tend to zero as the bond approaches maturity. This is not observed empirically, in fact, the term structure of credit spreads are observed to be bounded away from zero against the horizon  $T$ .

The Merton (1974) [94] model is straight-forward and non-complex in its hypothesis and it brings to the fore the underlying economic intuition of such a model and also implicitly serves to identify its inherent shortcomings. Although the structural model approach demonstrates important economic features, in itself, it has also formed the basis for further research to the extent the model is representative of observable real world economic phenomena. As a result we will identify features in the Merton (1974) [94] model and apply new assumptions to calibrate the model to observable market practices and characteristics.

### 3.3 The Model Setup

In this section we develop a first passage time model setup that is based on a generalization of the models postulated by Merton (1974) [94], Black and Cox (1976) [18] and Nielsen, Saá-Requejo and Santa-Clara (1993) [99], among others, and several of the assumptions are derived from these contributions. We suppose that the economy is complete and there are no arbitrage opportunities. The economy is defined over the time interval  $[0, T]$  where the distribution of the events is described by the filtered probability space  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0})$ . In addition, we assume the existence of an equivalent martingale measure  $\mathbb{Q}$  and all modelling is done relative to this risk-neutral measure. Formally, we construct the first-passage time model based on the assumptions below.

**Assumption 1:** The firm chooses its capital structure at time  $t$ . The value of the firm is designated as  $V$ . The choice of the capital structure consists of two components: risky debt,  $D$ , and equity,  $S$ . This combination of capital structure remains fixed without time limit until either (i) the firm's asset value falls to the default level or (ii) the debt matures. Moreover, we assume that there is a single issue of debt represented by a non-callable zero coupon bond of par value  $F$  and matures at time  $T$  in the future. The sum of the equity and debt add up to the total value of the firm and is expressed as:

$$V(t, T) = S(t, T) + D(t, T)$$

and the firm's value is the payoff in all states. This allows us to consider the firm's value as a traded security, alternately, the value of the firm is equal to the value of the assets of the firm.

The characterization of the capital structure into debt and equity components is an important assumption since we can apply the Black-Scholes (1973) [20] option pricing methodology to the economic modelling of credit risk and give insight to the approach that debt issued by a firm can be viewed as a contingent claim on the value of the assets of the firm. When the bond issue matures at time  $T$ , the bondholders will receive the par value  $F$  that is owing to them with the caveat that the value of the firm's assets is sufficient to honor this debt i.e.  $V_T > F$ , the equity owners then receive the balance of

the assets  $V_T - F$ . In particular, the time  $T$  value of the firm's equity maybe expressed as  $S_T = \max(0, V_T - F)$ . Similarly, at maturity,  $T$ , the payoff received by the bondholders maybe expressed as  $D_T = \min(V_T, F)$ . These payoff functions provides the key insight to value equity and debt as contingent claims on the firm's assets.

**Assumption 2:** As in Merton (1974) [94], Black and Cox (1976) [18] and Longstaff and Schwartz (1995) [91] the firm has productive assets whose value  $V$  follows a geometric Brownian motion:

$$dV = \mu V dt + \sigma V dB_t$$

where  $\mu$  is the total expected rate of return on the firm's assets;  $\sigma$  is the risk of the asset return and  $dB_t$  is the increment of a standard Brownian motion. The process  $V$  evolves continuously unless it breaches a default triggering barrier  $\bar{v}_t$ . We define  $\bar{v}_t$  shortly.

The diffusion process characterizes the value of the net cashflows generated by incremental shifts in the firm's economic activity that in turn translates into marginal changes in the firm's value. All cashflows are generated by productive processes and excludes cashflows arising from debt financing.

**Assumption 3:** We define the first-passage time of the firm value process  $V_t$  through the default threshold  $\bar{v}_t$  to be:

$$\tau = \inf \{t \in [0, T] : V_t \leq \bar{v}_t\}$$

We notice, that in the Merton (1974) [94] model market participants can observe the evolution of the continuous firm value  $V$ . Additionally, they observe the par value  $F$  and maturity  $T$  of the firm's risky debt. The default state can only be identified at maturity  $T$  when  $V_T < F$ . Similarly, in the Black and Cox (1976) [18] model market participants can observe the continuous firm value  $V$ . In contrast to the Merton (1974) [94] model default occurs at the first instant the firm value hits an observable threshold. In essence, the first passage time model of Black and Cox (1976) [18] is a generalization the Merton (1974) [94] model to allow for default prior to time  $T$  if the value of the firm falls to some pre-specified threshold  $\bar{v}_t$ .

Intuitively, this setup demonstrates the underlying appeal of structural models where default is deemed to be a predictable event, that is, by observing the firm value trajectory drifting close to the default threshold we anticipate default. Formally, we say the first passage time or default time  $\tau$  is a predictable stopping time. In technical terms, predictability can be described as the existence of an increasing sequence of stopping times  $(\tau_n)_{n \geq 1}$  such that  $\tau_n < \tau$  on  $\{\tau > 0\}$  and converges to  $\tau$  with probability one. This is precisely the situation for the first passage time assumption above.

**Assumption 4:** Following Black and Cox (1976) [18] we let  $\bar{v}_t$  be a pre-specified default threshold such that if the value of the firm's assets attains  $\bar{v}_t$  then default occurs at the instant  $V_t \leq \bar{v}_t$ . As long as the value  $V_t$  remains greater than  $\bar{v}_t$  the firm remains solvent

and is able to meet its debt obligation. We place an additional restriction on the default threshold such that default is only triggered when the value of the firm is worth less than an equivalent risk-free value. Consequently, the default threshold is specified as:

$$\bar{v}_t = \begin{cases} \gamma P(t, T) & \text{if } t < T \\ F & \text{if } t = T \end{cases} \quad (3.6)$$

where  $0 < \gamma \leq F$  and  $P(t, T)$  is the value of the riskfree bond.

The default boundary can be characterized by different specifications. For example, Longstaff and Schwartz (1995) [91] suppose that  $\bar{v}$  is constant through time and Black and Cox (1976) [18] show a  $\bar{v}_t$  as an exponential function of time. All these models make the implicit assumption that  $\bar{v}_t$  is the face value of the firm's liabilities and the ratio  $\frac{V_t}{\bar{v}_t}$  is observable which makes default predictable as  $\frac{V_t}{\bar{v}_t}$  increasingly approaches one.

Briys and de Varenne (1997) [23] argue that this type of specification of  $\bar{v}_t$  has some advantages. They indicate that in the presence of stochastic interest rates  $\bar{v}_t$  is also stochastic. Moreover, at default, the subsequent payoff to bondholders does not exceed the value of the firm. This serves to correct an apparent anomaly in the Longstaff and Schwartz (1995) [91] model. The untenable scenario in their model is that when the debt obligation matures the corporate can be in a solvent state relative to the default boundary but with lower value of assets than the mature bond value.

In general, we follow the definition of Black and Cox (1976) [18] to interpret the default boundary, that is, we view  $\bar{v}_t$  as the minimum value of the firm that is specified in the safety covenant of the debt contract for the firm to operate as a going concern. In the event  $V_t = \bar{v}_t$  we get a violation of the safety covenant which gives debt holders the right to force the firm into bankruptcy and claim ownership of the firm's assets.

In a corporate setting  $\bar{v}_t$  can have several economic interpretations. First, we can view  $\bar{v}_t$  as simply a measure of the firm's confidence to issue debt to finance its operations. Second,  $\bar{v}_t$  can be said to represent the conditional expected discount value of the default level  $\bar{v}_T$ . Finally, suppose a corporate increases its debt capacity by retiring equity all the while holding its assets and keeping its income constant. In corporate finance theory this is termed asset substitution. This increase in leverage portends proportional increase in risk for the equity holders since they have given the debt holders first claim on the corporate's assets and income. In turn the debt holders can constrain the high risk positions of the equity shareholders by imposing debt covenants which is exactly the purpose of  $\bar{v}_t$ . Implicit in the debt holders action is the assumption that the default threshold implies a decrease in debt capacity which makes the corporate to retire debt with equity financing. Consequently, the default threshold should trend with the firm value  $V_t$ . Adversely, any constant or decrease in firm value with a corresponding increase in  $\bar{v}_t$  we expect a prob-

able default.

**Assumption 5:** We specify the short-rate dynamics of the default free term structure. For now, we define a general formula for the short-rate process. Later we will specify particular parameterizations of the process under different model assumptions. To model the term structure of interest rates under the equivalent martingale measure  $\mathbb{Q}$  we consider a special case of the Vasicek (1977) [118] interest rate model. Let  $r$  be the short-term risk-free interest rate such that:

$$dr = \mu_r(r, t)dt + \sigma_r(r, t)d\tilde{B}_t$$

where  $\mu_r(r, t)$  and  $\sigma_r(r, t)$  are allowed to be non-constant parameters and  $\tilde{B}_t$  is a standard Brownian motion.

This type of interest rate process displays long term mean reverting dynamics. In addition, the characterization of the dispersion of interest rate changes in this model is said to follow a conditional normal distribution. The implication of this model specification is that interest rates can become negative. However, this drawback can be avoided if the model is calibrated properly and used appropriately. Nonetheless, this model is widely used because it is inherently highly tractable. This tractability is important in order to efficiently calibrate the bond pricing model.

More precisely, we consider the effect of the correlation between the changes in interest rates and the evolution of the firm's value on credit spreads. Longstaff and Schwartz (1995) [91] argue that an increase in the short-term interest rate when correlated positively with the risk-neutral drift of the firm value process reduces the probability of the firm value breaching the default threshold. The corresponding tightening in credit spreads is however contingent on the choice of the correlation coefficient. Several other theoretical models that show this correlation impact on credit spreads tend to explicitly allow for stochastic interest rates.

Interestingly, emerging empirical studies on two factor structural models show that models such as Longstaff and Schwartz (1995) [91] that incorporate stochastic interest rates and a correlation between firm value and interest rates has negligible empirical significance. One such comprehensive empirical study is by Eom et al. (2004) [46] where they argue that stochastic interest rates on average increase predicted credit spreads but these results are sensitive to the volatility estimates of the interest rate model. This insight will be crucial when we interpret the results from models presented in this chapter.

A motivating factor for continuing research in structural models is its apparent perception that the credit spreads generated by this class of models tend not to approximate those observed in the market. This drawback influenced researchers to introduce two factor models to investigate the impact of stochastic default free interest rates on the evolution of credit spreads.

**Assumption 6:** The payoffs to the different securities is contingent on the nature of the capital structure. In this model we assume that the capital structure comprise shares and risky bonds.

The bond payoff is their par value,  $F$ , in the event of no default. In the event of default the bond payoff is the difference in the fraction of the value of the firm and an arbitrary bankruptcy cost  $\kappa$ . The payoff function for bonds can be expressed as:

$$\bar{P}(V, t, T) = \begin{cases} \min\{F, V\} & \text{if } t = T \\ V - \kappa & \text{if } t < T \end{cases}$$

In the event of no default the shares payoff  $(V - F)^+$  at the maturity of the debt. We assume the model adheres to the strict absolute priority rule then the shares payoff nil in the event of default. The payoff function for shares can be expressed as:

$$S(V, t, T) = \begin{cases} \max\{V - F, 0\} & \text{if } t = T \\ 0 & \text{if } t < T \end{cases}$$

Clearly, in structural models, the pricing of credit risky bonds can only be accomplished if the component parts of the capital structure of the firm is completely specified. The value of both the equity and debt component is derived from the state variables which models the distribution of the firm's asset price process, the distribution of the interest rate and the apportionment of claims on the firm value in the event of default. We follow the traditional Merton (1974) [94] approach to value the component parts of the firm's capital structure as contingent claims.

In essence generic payoffs for the risky bonds and shares can be deduced from certain of the above assumptions, however, we impose strong restrictions on the default threshold. Each contingent claim is consistent with the characterization of the firm default process. In addition, we assume an efficient capital market with the qualification that the definition of default coincides with that of bankruptcy and these payoff functions come into effect either at (i) early default or (ii) at maturity of the bond obligation. In this context, at default, the firm's assets is never worth less than the par value of its debt.

The classic models of credit risk pricing treats corporate bonds as contingent claims whose payoff is derived from the total value of the assets of the firm. Moreover, these models explicitly suppose that the capital structure consists primarily of equity and a single zero coupon bond. While this setup commensurate with the no arbitrage option pricing methodology of Black-Scholes (1973) [20] and contrives to give elegant mathematical formulae it however mis-prices an important determinant of credit risk, the complexity of the capital structure.

A survey of the literature indicates that a large number of the researchers who build credit risk models have often overlooked pricing the credit risk of a corporate bond by explicitly modelling the capital structure of the obligor since the capital structure of firms are generally complex. For example, bonds itself can be callable, convertible, vary by tenure, coupon and priority.

This makes the resultant models complex and researchers may rely on numerical methods to compute bond prices. A common approach among researchers is to forego this capital structure complexity in favor of features that result in closed form solutions for bond prices. A second determinant of credit risk that arises from the complexity of the capital structure is the priority rights of claimants on the assets of the firm in the event of bankruptcy. In general, it is expected that the higher priority debt will take precedence over lower priority debt and this will directly affect the price of credit risk. Notwithstanding, in practice a strict adherence to priority rights may not always be the case, however, Altman et al. (2003) [1] show that recovery rates follow the expected priority rules.

We introduce a factor of bankruptcy costs in the payoff function of the credit risky bonds in the event of default. While, in practice bankruptcy costs is the first claim on the assets of the firm in our model it has the particular significance of ensuring that the payoff to the bondholders does not exceed the value of the firm at default.

In the next section, we turn our attention to the economic intuition underlying the  $T$ -forward risk neutral measure given this foreground setting of the first-passage time model.

### 3.3.1 The $T$ -Forward Risk Neutral Measure

In this section we aim to give an intuitive overview of the  $T$ -forward risk-neutral measure  $\mathbb{Q}^T$  and its relationship to the pricing of contingent claims. However, we first state the particular features that differentiates no-arbitrage pricing of contingent claims within the partial equilibrium framework of the Black-Scholes (1973) [20] model from the pricing of contingent claims with the term structure of the interest rates being driven by a stochastic process.

When pricing contingent claims in the Black-Scholes (1973) [20] framework we commonly define the money market account,  $B(t, T)$ , as the numéraire asset. The numéraire is at best defined as a deterministic function of time. In addition, if we price claims under the risk-neutral measure,  $\mathbb{Q}$ , then asset prices are martingales when discounted with the numéraire asset over the time interval  $[t, T]$ .

For an instant, if we give thought to the existence of no-arbitrage in an economy that has a complete term structure of bonds, that is, bonds with different maturities, as the primary traded financial instruments then we should sufficiently conclude that this can be achieved with the formulation of a well defined martingale measure. Recall, when working with zero-coupon bonds, it is not desirable to specify the bond price dynamics. Instead,



we work backwards by specifying the interest rate process and consequently, derive the bond price dynamics from the interest rates. Although short rate models of the diffusion type may still be used to derive bond price dynamics, they have a drawback with respect to the initial bond prices being outputs rather than inputs. For example, at time 0, we know the bond prices,  $P(0, T)$ , which can be observed in the market but the pre-defined diffusion type short-rate model may give initial bond prices that are different from current market prices. Subsequently, to get around this problem the researcher has to employ complex root search algorithms to approximate the best values of the drift and diffusion coefficients, respectively.

This phenomenon can be explained by the observation the short-rate is not the only interest rate available in the market, that is, bonds of different maturities are associated with correspondingly different interest rates. As an example, we note that the interest rate on a loan for 6 months is markedly different from the interest rate on a loan to be settled over 2 years. Hence a preferred model of interest rates should include all the information about the different rates associated with different maturities. We can accomplish this if we shift our focus from the short-rate to forward rates that allow initial bond prices to be inputs.

Now, in the context of interest rate theory we can intuitively interpret the forward rate  $f(t, T)$  as the interest rate that applies over the infinitesimal time interval  $[T, T + dT]$  which can be locked in at time  $t$ . If the bond prices are sufficiently smooth we can define the forward rate as:

$$f(t, T) = -\frac{\partial \ln P(t, T)}{\partial T} \quad (3.7)$$

Then, at time  $t$  we have a continuum of forward rates, that is, one for each maturity  $T > t$ , and we can enhance this concept if we can view the continuum of forward rates as a forward curve. We can invert (3.7) to express bond prices in terms of forward rates as:

$$P(t, T) = \exp \left( - \int_t^T f(t, u) du \right)$$

and is valid under no arbitrage conditions. Further, if we can invest in a risk-less asset at time  $t$  then the return will accrue at a rate  $f(t, u)$  at time  $u \geq t$ . Notice, that  $f(t, t) = r_t$  is just the short-rate so that specifying the forward rate also determines the dynamics of the short-rate. We have now shown that the forward rate is inextricably linked to both the short-rate of interest and the bond price dynamics.

Thus we know that just as the money market account,  $B(t, T)$ , represented as a deterministic function of time and defined under an equivalent martingale measure is a suitable numéraire asset in the Black-Scholes (1973) [20] framework. Similarly, we claim that with stochastic interest rates in the Heath-Jarrow-Morton (1992) [63] framework characterized by a well-defined risk-neutral probability measure makes for discounted asset prices to be martingales. It turns out that by carefully defining risk-neutral probability measures we are able to solve various contingent claim formulae while at the same time the equivalent

martingale measure is dependent on the prudent choice of the numéraire. Therefore it is fair to say that changing probability measures is not independent of the numéraire asset. Recall, we pointed out above that the short-rate of interest is implicitly related to bond prices. It turns out that under the general framework of the change of numéraire concept postulated by Geman, Karoui and Rochet (1995) [53] that, for example, the bond price  $P(t, T)$  can be a suitable numéraire asset in a model. Then with the new numéraire asset being a risk-free zero-coupon bond, the equivalent martingale measure,  $\mathbb{Q}^T$ , is often times referred to as the  $T$ -forward risk-neutral measure since financial security prices valued in terms of the bond prices have an economic interpretation of forward prices. Put differently, under the forward risk-neutral measure forward prices are fair approximations of future market prices.

With this overview of the  $T$ -forward risk-neutral measure we are now able to employ this tool to the formulation of specific contingent claims.

### 3.4 Stochastic Default Boundary Model

The historical research on structural models emphasized a conceptual framework to relate credit risk to the financial economics of the firm processes. For example, the Merton (1974) [94] model was extended to quantify the effect of varying debt maturity, the impact of leverage, adding tax and so forth. Cumulatively, much of this research continued to incorporate the constant interest rate assumption albeit we observe that market phenomena reveal that interest rates decidedly exhibit a term structure.

For investors in a firm, the pricing of credit sensitive financial instruments, for example, mortgage-backed securities or floating rate corporate bonds, the specification of credit risk is inherently associated with the term structure of interest rates. On the other hand, the economic activity of the firm and its underlying capital structure covenants may also be implicitly exposed to interest rates. Thus for firm value models to be quantified more precisely it would be prudent to define a correlation between the credit risk process and the process driving the term structure of interest rates. Notwithstanding, subsequent research has shown that there is a clear dependence between the pricing of credit risk and the term structure of interest rates. Shimko et al. (1993) [114] produced one of the earliest such studies and in their work they implement a generalization of the Merton (1974) [94] model to price credit risk with the assumption of stochastic interest rates.

In this section we present a theoretical model of credit risk as was first shown by Schönbucher (2000) [112]. This is a continuous time first passage time model with geometric Brownian motion dynamics and driven by a stochastic term structure of interest rates of the type postulated by Vasicek (1977) [118]. The exponential default threshold is chosen judiciously so that the payoff to bondholders is always less than the value of the firm at default. We anticipate violation of the absolute priority rule and we choose

exogenously a fraction of the firm value as bankruptcy costs such that the payoff at default is the difference in the fraction of the value of the firm and bankruptcy costs. By considering that the risk-neutral drift of a defaultable bond must be exactly the risk-free rate of interest, we derive the arbitrage free fundamental pricing equation for defaultable bonds. Finally, as a special case we adopt the Briys and de Varenne (1997) [23] first passage time model approach to derive a closed form solution to the defaultable zero coupon bond pricing equation.

### 3.4.1 Firm Value Model

The firm's capital structure comprises of equity and debt with cumulative value denoted by  $S$  and  $D$ , respectively. To be explicit in our exposition we formally adopt the following definitions:

- $V_t$  = value of the firm's assets at time  $t \in [0, T]$
- $D_t$  = value of the firm's debt at time  $t \in [0, T]$
- $S_t$  = value of the firm's equity at time  $t \in [0, T]$

Now, at every time  $t \in [0, T]$  we observe that

$$V_t = S_t + D_t$$

The traded securities are a share,  $S_t$ , and a defaultable zero-coupon bond,  $\bar{P}_t$ , of par value  $F$  redeemable at maturity  $T$ .

Pricing under the risk-neutral probability measure  $\mathbb{Q}$  we define the dynamics of the firm value process to evolve according to a geometric Brownian motion with:

$$dV = rVdt + \sigma V d\check{B}_t \quad (3.8)$$

where  $r$  is the drift of the process and is equal to the risk neutral rate of return. Further, under risk-neutral dynamics we define a one factor short-rate model to evolve such that:

$$dr = \mu_r(r, t)dt + \sigma_r(r, t)d\tilde{B}_t \quad (3.9)$$

with  $\tilde{B}_t$  a standard Brownian motion. In addition, we assume there exists a correlation between the dynamics of the value of the firm and the dynamics of the interest rates such that we define

$$Cov(d\check{B}_t, d\tilde{B}_t) = \rho dt$$

where  $\rho$  is the instantaneous correlation coefficient between  $d\check{B}_t$ , the Brownian motion driving the firm's value, and  $d\tilde{B}_t$ , the Brownian motion driving the interest rates.

A safety covenant allows bondholders to protect their interests in the firm in the event of a default. In this model we specify the safety covenant as an exogenous, time-dependent

lower boundary  $\bar{v}_t$  as per (3.6) (see Assumption 4). Notice, that the safety covenant is defined such that the payoff to the bondholder at default does not exceed the par value of the outstanding liability. The first passage-time for the firm value process  $V_t$  to access the default threshold  $\bar{v}_t$  is:

$$\tau = \inf \{t \in [0, T] : V_t \leq \bar{v}_t\}$$

and  $\tau$  is commonly referred to as the default time. To complete the model specification we modify an original assumption, that is, we incorporate the real-world observation that the market anticipates a deviation from the absolute priority rule and specify this as:

$$\kappa = \tilde{\kappa}P(t, T)$$

where  $\kappa$  is the bankruptcy costs and is represented as a fraction of the price of a risk-free bond with the caveat that the advantage is decided in favor of the shareholders.

Recall, from a global perspective we are still in the structural model framework. This implicitly asserts that the components of the firm's capital structure can be modelled as contingent claims on the value of the firm's assets. As such we consider the share,  $S$ , and the risky bond,  $\bar{P}$ , as contingent claims on the value of the firm,  $V_t$ . If there is no default then all payoffs are as per contract. Additionally, for ease of exposition we specify that the share,  $S$ , the risky bond,  $\bar{P}$ , and the constant  $\gamma$ , associated with the default threshold, be normalized to 1. As usual, the default payments are triggered at the default threshold  $\bar{v}_t$ . The final payoff of the share,  $S$ , is expressed as:

$$S(t, T) = \begin{cases} (V - 1)^+ & \text{if } no \text{ default} \\ \tilde{\kappa}P(t, T) & \text{if } default \end{cases}$$

The final payoff for the risky bonds is expressed as:

$$\bar{P}(t, T) = \begin{cases} 1 - (1 - V)^+ & \text{if } no \text{ default} \\ P(t, T)(1 - \tilde{\kappa}) & \text{if } default \end{cases}$$

The payoffs are subject to the condition that there are no cash outflows during the period the bond debt is outstanding. This means that the firm is neither allowed to repurchase any equity nor issue any senior or equal priority debt on the firm's assets. Similar to the Merton (1974) [94] model we treat all securities as contingent claims on the assets of the firm.

### 3.4.2 Pricing a Credit Risky Bond

The first passage time model is now completely specified in the classic firm value framework and as such we can now apply the standard option theoretic methodology to price credit risky debt. In particular, we determine the pricing formula for a defaultable zero-coupon bond. In general, a defaultable zero-coupon bond is a financial security

promising to pay 1 unit of currency at some maturity date,  $T$ , in the future.

We assume that the underlying probability space  $(\Omega, \mathcal{F}, \mathbb{Q})$  equipped with the filtration  $(\mathcal{F}_t)_{t \geq 0}$  is rich enough to support the short-rate process  $r$ . When we consider the modelling of defaultable claims associated with stochastic dynamics it is fair to assume that both the riskfree interest rate and the firm value are state variables. Then by Itô's lemma we can derive the bond dynamics as:

$$\begin{aligned} d\bar{P} &= \left[ \frac{\partial \bar{P}}{\partial t} + \frac{1}{2} \sigma^2 V^2 \frac{\partial^2 \bar{P}}{\partial V^2} + \rho \sigma_r \sigma V \frac{\partial^2 \bar{P}}{\partial r \partial V} + \frac{1}{2} \sigma_r^2 \frac{\partial^2 \bar{P}}{\partial r^2} \right] dt \\ &+ \frac{\partial \bar{P}}{\partial V} dV + \frac{\partial \bar{P}}{\partial r} dr \end{aligned} \quad (3.10)$$

We assert that the risk-neutral drift of the defaultable bond must equal  $r\bar{P}dt$ . Intuitively, we know that if the stochastic component in the formula for the defaultable bond dynamics vanish then we are just left with a deterministic formula, that is, we have eliminated one level of complexity from the pricing equation. As such we equate the risk-neutral drift,  $r\bar{P}dt$ , to the drift terms in (3.10) to get:

$$\begin{aligned} r\bar{P}dt &= \left[ \frac{\partial \bar{P}}{\partial t} + \frac{1}{2} \sigma^2 V^2 \frac{\partial^2 \bar{P}}{\partial V^2} + \rho \sigma_r \sigma V \frac{\partial^2 \bar{P}}{\partial r \partial V} + \frac{1}{2} \sigma_r^2 \frac{\partial^2 \bar{P}}{\partial r^2} \right] dt \\ &+ rV \frac{\partial \bar{P}}{\partial V} dt + \mu_r \frac{\partial \bar{P}}{\partial r} dt \end{aligned}$$

where  $\bar{P}$  is a solution to the fundamental pricing equation:

$$\begin{aligned} 0 &= \frac{\partial \bar{P}}{\partial t} + \frac{1}{2} \sigma^2 V^2 \frac{\partial^2 \bar{P}}{\partial V^2} + \rho \sigma_r \sigma V \frac{\partial^2 \bar{P}}{\partial r \partial V} + \frac{1}{2} \sigma_r^2 \frac{\partial^2 \bar{P}}{\partial r^2} \\ &+ rV \frac{\partial \bar{P}}{\partial V} + \mu_r \frac{\partial \bar{P}}{\partial r} - r\bar{P} \end{aligned} \quad (3.11)$$

for a defaultable zero-coupon bond subject to the following conditions:

$$\bar{P}(t, T) = \begin{cases} 1 - (1 - V)^+ & \text{if } no \text{ default} \\ P(t, T)(1 - \tilde{\kappa}) & \text{if } default \end{cases}$$

such that no-default represents the final condition and default represents the boundary condition, respectively.

### 3.4.3 Special Case: Briys and de Varenne Solution to the Credit Risky Bond Equation

The aim of this section is to postulate a solution for the zero-coupon defaultable bond equation as in (3.11). To ensure the existence of a closed form solution we adopt as a

special case the Briys and de Varenne (1997) [23] solution to the fundamental pricing equation. In fact, the Briys and de Varenne (1997) [23] model is a special case of the Black and Cox (1976) [18] model albeit with stochastic interest rates. As the next step we specify that the price of the defaultable zero-coupon bond can be represented under the  $T$ -forward risk-neutral measure. Further, we derive the forward firm value dynamics and show that the diffusion coefficient to be a deterministic function of time. Consequently, we apply the property of a random time-change for Brownian motion to eliminate the time-dependence of the diffusion coefficient. Finally, we calculate the probability of default which completely specifies the Briys and de Varenne (1997) [23] solution zero-coupon defaultable bond equation.

It is prudent to place ourselves in the Gaussian Heath-Jarrow-Morton (1992) [63] setup. This means that we can work under the  $T$ -forward risk-neutral measure. We proceed with the solution under the following assumptions:

1. The bond price volatility is a deterministic function of time.
2. The default triggering threshold equals  $\bar{v}_t = \gamma P(t, T)$  for some constant  $\gamma$ .

Briys and de Varenne (1997) [23] model the short rate process as a version of the Vasicek (1977) [118] model with the risk-free rate volatility a deterministic function. The immediate consequence is that the risk-neutral price of the default-free zero coupon bond with maturity  $T$  can be expressed as:

$$dP = rPdt + \sigma_r(T - t)PdB_t^1 \quad (3.12)$$

In addition, the dynamics of the firm's value process is modelled by a stochastic differential equation of the form:

$$dV = rVdt + \sigma_v V(\rho dB_t^1 + \sqrt{1 - \rho^2} dB_t^2) \quad (3.13)$$

where  $\rho$  is the correlation coefficient between the value of the firm and the risk-less interest rates,  $\sigma_v$  the volatility of the firm's value and  $B_t^1$  and  $B_t^2$  are two uncorrelated Brownian motions.

With the intuitive overview of the  $T$ -forward risk-neutral measure,  $\mathbb{Q}^T$ , described under the model setup we now employ this construct to the evaluation of the fundamental pricing equation. Consequently, we can express the price of the defaultable zero-coupon bond  $\bar{P}(t, T)$  under the  $T$ -forward risk-neutral measure as:

$$\bar{P}(t, T) = P(t, T) - \kappa P(t, T) \mathbb{Q}^T[\tau < T] \quad (3.14)$$

which is basically the difference between the price of a default-free bond and the product of probability of default, the price of a default free bond and the bankruptcy costs at default,  $\kappa$ . But we know that the sum of the probability of default,  $\mathbb{Q}_D^T[\tau < T]$ , and the probability of survival  $\mathbb{Q}_S^T[\tau \geq T]$  must equal one, such that:

$$\bar{P}(t, T) = P(t, T) \left( 1 - \kappa (1 - \mathbb{Q}^T[\tau \geq T]) \right) \quad (3.15)$$

The event  $\{\tau \geq T\}$  is only valid if the default threshold is not reached during the time interval  $[t, T]$ . The *forward value* of the firm is defined as  $\tilde{V} =: \frac{V_t}{P(t, T)}$  and has dynamics under the  $T$ -forward risk-neutral measure  $\mathbb{Q}^T$ :

$$\begin{aligned} \frac{d\tilde{V}}{\tilde{V}} &= (\rho\sigma_v - \sigma_r(T-t))dB_1^T + \sigma_v\sqrt{1-\rho^2}dB_2^T \\ &=: \sigma(t)d\hat{B} \end{aligned} \quad (3.16)$$

and the deterministic volatility  $\sigma(t)$  is expressed as:

$$\sigma(t) = \left[ \sigma_v^2 - 2\rho\sigma_v\sigma_r(T-t) + \sigma_r^2(T-t)^2 \right]^{\frac{1}{2}} \quad (3.17)$$

We construct  $\hat{B}_t$  as a standard Brownian motion from  $B_1^T$  and  $B_2^T$ . We have now transformed our original Briys and de Varenne (1997) [23] type dynamics of the firm value process to a forward firm value process denoted as  $\tilde{V}_t$ , modelled under the  $T$ -forward risk-neutral measure that again follows the dynamics of a standard Brownian motion and associated with a time-dependent diffusion coefficient. In addition, the key to the zero-coupon defaultable bond pricing formula is to calculate the probability of the event  $\{\tau \geq T\}$ . In the next section we will apply the fact that a time-change of an Itô process is again an Itô process to advance toward a solution of the bond pricing formula.

### 3.4.3.1 Random Time Change

Notice that the dynamics of the forward firm value process is defined in terms of a time-dependent volatility stochastic process with the diffusion coefficient expressed as in (3.17). We apply the notion of a random time change (see Øksendal (1998) [101]), in particular, the result that allows us to recognize a time change for a stochastic process is again a stochastic process albeit driven by a different Brownian motion. This allows us to eliminate the deterministic feature in the diffusion coefficient.

To effect the random time-change we define the process  $X_t$  to be expressed as:

$$dX_t = \sigma(t)d\hat{B}_t$$

and the quadratic variation of the time-changed process is given by  $\langle X \rangle_t =: \nu(t)$  where  $\sigma(t)$  is referred to as the *time change rate*.

Now, if we apply Theorem 4.6 from Karatzas and Shreve (1991) [78] to transform the process  $X_t$  at time  $t$  to a time-changed Brownian motion, then we get a new process represented as the value of a standard one-dimensional Brownian motion  $Z_{\nu(t)}$  at time  $\nu(t)$  that can be expressed as  $X_t = Z_{\nu(t)}$ . In a similar manner we can define the time

change for the forward firm value process  $\tilde{V}_t$  to be  $H_{\nu(t)} = \tilde{V}_t$ . Next, we apply Proposition 4.8 from Karatzas and Shreve (1991) [78] for time-change for stochastic integrals to the forward firm value process  $\tilde{V}$  to get:

$$\tilde{V}_t = \int_0^t \tilde{V}_s dX_s = \int_0^{\nu(t)} H_r dZ_r = H_{\nu(t)} \quad (3.18)$$

From the right-hand side of (3.18) we notice that  $H$  satisfies the stochastic differential equation  $dH_t = H_t dZ_t$  and with the random time change property for Brownian motion we have transformed the deterministic feature of the volatility function to a constant quantity. Similarly, we have applied the time change concept to transform the forward firm value process to a SDE with no drift. The random time change transforms aids the solution of fundamental pricing equation and its implications will be clearer in the next section. We can now apply standard techniques to solve for the probability of default under the  $T$ -forward risk-neutral measure.

### 3.4.3.2 The Probability of Default

We are ready to turn our attention to the event described by  $\{\tau \geq T\}$  and hence calculate the probability that the forward firm value trajectory will reach the default threshold. The event that the time of default,  $\tau$ , is greater than or equal to the defaultable bond maturity date,  $T$ , is otherwise described as  $\{\tilde{V} \geq \gamma\}$  for  $t \leq T$ . Then if we match the variables in this event to the time-changed variables we have  $\{H_t \geq \gamma\}$  for  $t \leq \nu(T)$  and define  $H_0$  as the initial value of the time-changed process  $H_t$ .

The next step is to rewrite  $\ln H_t = Z_t$  in terms of a new variable  $m$  such that the new process resembles a standard diffusion process. In addition, we define  $\omega := \ln H_t$ .

The probability distribution function for a general running minimum process of this type is commonly tabulated. Hence we apply Corollary B.3.4 (see Musiela and Rutkowski (1997) [97]) to the event defined as  $\{H_t \geq \gamma\}$  for  $t \leq \nu(T)$  to calculate the probability of default to be:

$$\mathbb{Q}^T\{m_t^\omega\} = N(x_1) - e^{-2y} N(x_2)$$

where

$$x_1 = \frac{y + \frac{1}{2}\nu(T)}{\sqrt{\nu(T)}}$$

$$x_2 = \frac{-y + \frac{1}{2}\nu(T)}{\sqrt{\nu(T)}}$$

and  $y = \ln H_0 - \ln \gamma$  is the default threshold. With this result we have now completely solved the zero coupon defaultable bond pricing equation.



### 3.4.3.3 A Closed Form Solution

The probability of default equation as shown in the previous section now enables us to summarize the main result for the Briys and de Varenne (1997) [23] solution for the price of a zero-coupon defaultable bond. Recall, the price of a default risky zero-coupon bond was represented as:

$$\bar{P}(t, T) = P(t, T) \left( 1 - \kappa (1 - \mathbb{Q}^T[\tau \geq T]) \right)$$

where  $P(t, T)$  is the riskfree zero coupon bond and  $\mathbb{Q}^T$  is just the  $T$ -forward risk neutral measure and the probability of default is calculated as :

$$\mathbb{Q}^T[\tau \geq T] = N(x_1) - e^{-2y} N(x_2)$$

where

$$x_1 = \frac{y + \frac{1}{2}\nu(T)}{\sqrt{\nu(T)}}$$

$$x_2 = \frac{-y + \frac{1}{2}\nu(T)}{\sqrt{\nu(T)}}$$

with  $y$  and  $\nu(t)$  are expressed as:

$$y = \ln \frac{V_0}{P(0, T)\gamma} \quad (3.19)$$

$$\nu(T) = \sigma_v^2 T - \rho \sigma_v \sigma_r T^2 + \frac{1}{3} \sigma_r^2 T^3 \quad (3.20)$$

We now have a complete term structure of defaultable zero-coupon bonds.

## 3.5 A Short Note on the Default Boundary

The generalized first passage time model can be modified to encompass several valuation problems as special cases. For the purposes of this dissertation we will treat two cases that can be described as:

- the short-rate of interest is defined as constant, that is,  $r_t = r$  for all  $t \geq 0$  and a constant default boundary  $\bar{v}$ .
- for a constant  $\gamma$ , let the threshold function be defined as a deterministic default boundary such that  $\bar{v}(t) = \beta e^{-\gamma(T-t)}$ .

A primary assumption in structural credit risk modelling is the specification of the default triggering mechanism. In this study we characterize the default triggering mechanism as a lower threshold and generally term this as the *default boundary*. In particular, we follow the traditional modelling of the default boundary as an exogenously specified

covenant and that is normally a function of the par value single-issue zero coupon bond. More specifically, we assume default occurs at the first passage time the market value of the corporate's assets breaches the default boundary. While the stochastic default boundary model is analytically constructed to capture empirical features from corporate finance theory, for example, bankruptcy costs, deviation from absolute priority, stochastic interest rates and the like such a general model implicitly embeds other models that specialize this framework. We show two such models, a constant- and a deterministic default boundary model. As a result the natural consequence of varying the default mechanism is different prices of credit risky bonds, levels of credit spreads and default probabilities.

There are several reasons we choose a differing default boundary. A stochastic default boundary can reflect the nature of the capital structure of the corporate. First, the corporate can have a complex capital structure, for example, options, derivatives and perpetual debt and the associated payoffs on these financial instruments could be contingent on unhedged risks and unpredictable cashflows. Second, we consider how the investor's beliefs about the corporate's risk structure has evolved through time with the belief that if the corporate is high risk then it has lower profitability. This could occur if the corporate is relatively new or has been in existence for a short period and as such has a limited history. We interpret this as a severe information asymmetry problem with little or no scope for the investor to update his beliefs. Third, following Jensen and Meckling (1976) [75] there could be conflicts of interest between equity holders and managers since managers only capture a proportion of the gain from their profit generating projects. This is the classic problem of agency costs and if the conflict is not sufficiently mitigated then this can adversely effect the corporate's profit enhancing activities. Finally, relatively new corporates may issue long-term debt and investor's may demand a higher premium for investing in its projects since they remain unsure about the long-term outlook of the corporate's cashflow predictability. As a result the corporate may experience increased short-term financing costs that could impact on its debt servicing obligations. Cumulatively these factors show a probable randomness in the corporate's income streams. Consequently, a stochastic default boundary can capture the effects of these shocks on a corporate's activities.

We can similarly characterize the deterministic default boundary model to reflect the capital structure of the corporate. First, the corporate can have a less complex capital structure, for example, only debt, albeit of varying tenures and with the income streams of the corporate generally predictable. Second, in comparison to a relatively new corporate we consider a corporate that has been operating for a longer period with investors having increased but not complete private information. We interpret this as a partial information asymmetry problem with discrete opportunities for the investor to update his beliefs. Third, the corporate has more intermediate than long-term debt with greater information asymmetry associated with the long-term debt. This is likely since long tenure debt obligations may be contingent on the quality of the skills of present and future management. On the other hand, intermediate cashflows may be determined by completed activities and associated with which there is little information asymmetry. As a result, investors

have updated beliefs with respect to short-term debt and accept lower relative premiums for intermediate tenure debt that has the impact of reduced debt financing obligations. Taken together these factors reflect that the corporate is generally predictable in the short-term and is less predictable in the long-run and as such we suppose a monotone increasing default boundary to capture the effects of the corporate's activities.

Like the Merton (1974) [94] and the Longstaff and Schwartz (1995) [91] models, we consider a model with a constant default boundary. First, the corporate can have a simple capital structure, for example, a single issue of debt and with the cashflows of the corporate generally predictable. In addition, the assumption of a single issue of debt is consistent with a stationary capital structure. Second, we consider the corporate to be a well-established entity that has been in existence for a long period and has a documented record of its history. Corporates in this category generally opt to engage in positive net present value projects. While we cannot abstract ourselves absolutely from information asymmetry effects investors deem this type of corporate to be of relative lower risk and a preferred investment entity. Third, with a long company history investors maybe better positioned to assess the risk of the corporate's future cash streams and may generally regard this exercise as predictable. In addition, investment premiums tend to be the lowest for these types of corporates. Consequently, we assert that these characteristics of the corporate can be shown by a constant default boundary.

### 3.6 Constant Default Boundary Model

Formally, we present the following assumptions that are modified from the original set of assumptions postulated in the model setup. All other assumptions remain unchanged.

**Assumption 1:** In general, we let  $\bar{v}$  be a pre-specified default threshold such that if the value of the firm's assets reaches  $\bar{v}$  then default occurs at the instant:

$$V_t \leq \bar{v}$$

where  $V_t$  is the firm value process.

**Assumption 2:** Consequently, we define the first-passage time of the firm value process  $V_t$  through the default threshold  $\bar{v}$  to be:

$$\tau = \inf \{t \in [0, T] : V_t \leq \bar{v}\}$$

**Assumption 3:** We define the risk-less spot rate of interest to be the process  $(r_t)_{t \geq 0}$ , but for modelling purposes we suppose that  $r_t = r$  is constant for all time  $t$ , then the price of a default risk free zero coupon maturing at time  $t$  is  $P(t) = e^{-rt}$ .

**Assumption 4:** The bond payoff is their par value,  $F$ , in the event of no default. In the event of default the bond payoff is the value of the firm at the instant of default. The payoff function for risky bonds can be expressed as:

$$\bar{P}(V, t, T) = \begin{cases} \min\{F, V\} & \text{if } t = T \\ V & \text{if } t < T \end{cases}$$

A safety covenant allows bondholders to protect their interests in the firm in the event of a default. In this model we specify the safety covenant as an exogenous constant lower boundary  $\bar{v}$ , where  $\bar{v}$  satisfies  $\bar{v} < V_0$  with  $V_0$  being the initial value of the firm value process. If there is no default then all payments are as per contract else the default payments are triggered at the default threshold  $\bar{v}$ .

### 3.6.1 Pricing a Credit Risky Bond

By Itô's lemma we can derive the bond dynamics as:

$$d\bar{P} = \left[ \frac{\partial \bar{P}}{\partial t} + \frac{1}{2} \sigma^2 V^2 \frac{\partial^2 \bar{P}}{\partial V^2} \right] dt + \frac{\partial \bar{P}}{\partial V} dV \quad (3.21)$$

We assert that the risk neutral drift of the defaultable bond must equal  $r\bar{P}dt$ . As such we equate the risk neutral drift  $r\bar{P}dt$  to the drift terms in (3.21) to get:

$$r\bar{P}dt = \left[ \frac{\partial \bar{P}}{\partial t} + \frac{1}{2} \sigma^2 V^2 \frac{\partial^2 \bar{P}}{\partial V^2} \right] dt + rV \frac{\partial \bar{P}}{\partial V} dt$$

where  $\bar{P}$  is a solution to the fundamental pricing equation:

$$0 = \frac{\partial \bar{P}}{\partial t} + \frac{1}{2} \sigma^2 V^2 \frac{\partial^2 \bar{P}}{\partial V^2} + rV \frac{\partial \bar{P}}{\partial V} - r\bar{P} \quad (3.22)$$

We observe that expression (3.22) is nothing more than the Black-Scholes PDE. However, the solution to (3.22) is contingent on the payoff conditions of the credit risky bond. Now, under the assumptions of constant interest rate  $r$  and constant default threshold we can express the time  $t$  price of a credit risky zero coupon bond in terms of the generic formulation:

$$\bar{P}(t, T) = \mathbb{E}_{\mathbb{Q}} \left[ e^{-r(T-t)} \left( FI_{\{\tau > T\}} + \bar{F} I_{\{\tau \leq T\}} \right) \right] \quad (3.23)$$

where  $\mathbb{E}_{\mathbb{Q}}(\cdot)$  denotes the expectation under the probability measure  $\mathbb{Q}$  and  $I_A$  is the indicator function of  $A$ . Additionally, we can express (3.23) in terms of the payoff conditions such that:

$$\bar{P}(t, T) = \mathbb{E}_{\mathbb{Q}} \left[ e^{-r(T-t)} \left( FI_{\{\tau > T\}} + \bar{v} I_{\{\tau \leq T\}} \right) \right]$$

where  $F$  is the par value of the bond and is redeemed in the event of no default. Alternately, in the event of default the value of the bond is just equal to the value of the default threshold,  $\bar{F} = \bar{v}$ . Following the general formulation for the default probability  $\mathbb{Q}(\tau \leq T)$  from Bielecki and Rutkowski (2002) [14] the price of a defaultable zero coupon bond can be expressed as:

$$\bar{P}(t, T) = e^{-r(T-t)} F \left[ 1 - \mathbb{Q}(\tau \leq T) \right] + \bar{v} e^{-r(T-t)} \mathbb{Q}(\tau \leq T)$$

where

$$\begin{aligned} \mathbb{Q}(\tau \leq T) &= N(x_1) + \left( \frac{\bar{v}}{V_t} \right)^{2a} N(x_2) \\ x_1 &= \frac{\ln \frac{\bar{v}}{V_t} - \left( r - \frac{1}{2} \sigma_V^2 \right) (T-t)}{\sigma_V \sqrt{T-t}} \\ x_2 &= \frac{\ln \frac{\bar{v}}{V_t} + \left( r - \frac{1}{2} \sigma_V^2 \right) (T-t)}{\sigma_V \sqrt{T-t}} \\ a &= \frac{r - \frac{1}{2} \sigma_V^2}{\sigma_V^2} \end{aligned}$$

and  $N(\cdot)$  is the cumulative normal distribution. For ease of implementation we rewrite the credit risky bond price formula as shown in Appendix 3.1.

In this special case of the stochastic default boundary model we make the simplifying assumption that the short term interest rate is constant and is equal to  $r$ . Default occurs when the firm value process hits a lower constant default threshold. From a technical perspective we derive the credit risky bond value dynamics and state the price of a defaultable bond to be a solution to this PDE. We show that the PDE admits a closed form solution. In addition, we notice that the risk neutral default probability is of the type commonly used in pricing applications of barrier options. An economic interpretation of this model is that it describes a financial contract such that it pays a recovery value of cash at time  $t$  if the underlying firm value  $V_t$  reaches the barrier  $\bar{v}$  before time  $T$ . This precisely describes an exotic contingent claim of a European Down-and-In Cash-or-Nothing type option where the amount of cash at default is the recovery value.

### 3.7 Deterministic Default Boundary Model

A second variant of the stochastic default boundary model closely resembles the work of Black and Cox (1976) [18]. In this model we make assumptions of constant risk free interest rate and a deterministic default boundary.

A safety covenant is a mechanism that allows the bondholders to file for bankruptcy if a firm performs less than expected relative to a set benchmark. In practice, it is common to find safety covenants that issue a right to bondholders to immediately demand the entire amount of the outstanding debt issue if the debt issuer is not able to meet coupon payments or principal obligations timeously. The implication of this action by bondholders is to allow the firm an opportunity to restructure its debt or induce bankruptcy. Thus

the effect of the safety covenant is to protect the bondholders interests against undue devaluation of the firm's assets.

Similar to the Black and Cox (1976) [18] model we propose an exogenous, deterministic boundary as a safety covenant. In particular, we introduce a lower boundary,  $\bar{v}_t$ , at which bondholders will declare bankruptcy as soon as the value of the assets of the firm reaches this lower boundary. This boundary is exponentially distributed and is represented as:

$$\bar{v}_t = \beta e^{-\gamma(T-t)}$$

for  $t \in [0, T)$  where  $\beta$  and  $\gamma$  are given as exogenous constants. In this model as the firm value passes the default barrier at time  $t$  the firm is instantaneously in default of its debt obligation.

Formally, we present the following assumptions that are modified from the original set of assumptions postulated in the model setup. All other assumptions remain unchanged.

**Assumption 1:** In general, we let  $\bar{v}_t$  be a pre-specified default threshold such that if the value of the firm's assets reaches  $\bar{v}_t$  then default occurs at the instant:

$$V_t \leq \bar{v}_t$$

where  $V_t$  is the firm value process.

**Assumption 2:** Consequently, in this model we define the first-passage time of the firm value process  $V_t$  as the first instant the safety covenant is breached to be:

$$\tau = \inf \{t \in [0, T] : V_t \leq \bar{v}_t\}$$

**Assumption 3:** We define the risk-less spot rate of interest to be the process  $(r_t)_{t \geq 0}$ , but for modelling purposes we suppose that  $r_t = r$  is constant for all time  $t$ , then the price of a default risk free zero coupon maturing at time  $t$  is  $P(t) = e^{-rt}$ .

**Assumption 4:** The bond payoff is the par value,  $F$ , in the event of no default. In the event of default the bond payoff is the value of the firm at the instant of default. The payoff function for risky bonds can be expressed as:

$$\bar{P}(V, t, T) = \begin{cases} \min\{F, V\} & \text{if } t = T \\ V & \text{if } t < T \end{cases}$$

### 3.7.1 Pricing a Credit Risky Bond

By Itô's lemma we can derive the bond dynamics as:

$$\bar{P} = \left[ \frac{\partial \bar{P}}{\partial t} + \frac{1}{2} \sigma^2 V^2 \frac{\partial^2 \bar{P}}{\partial V^2} \right] dt + \frac{\partial \bar{P}}{\partial V} dV \quad (3.24)$$

We assert that the risk neutral drift of the defaultable bond must equal  $r\bar{P}dt$ . As such we equate the risk neutral drift  $r\bar{P}dt$  to the drift terms in (3.24) to get:

$$r\bar{P}dt = \left[ \frac{\partial \bar{P}}{\partial t} + \frac{1}{2}\sigma^2 V^2 \frac{\partial^2 \bar{P}}{\partial V^2} \right] dt + rV \frac{\partial \bar{P}}{\partial V} dt$$

where  $\bar{P}$  is a solution to the fundamental pricing equation:

$$0 = \frac{\partial \bar{P}}{\partial t} + \frac{1}{2}\sigma^2 V^2 \frac{\partial^2 \bar{P}}{\partial V^2} + rV \frac{\partial \bar{P}}{\partial V} - r\bar{P} \quad (3.25)$$

with boundary conditions:

$$\begin{aligned} \bar{P}(V, T, T) &= \min(V, F) \\ \bar{P}(\bar{v}_t, t, T) &= \beta e^{-\gamma(T-t)} \end{aligned}$$

The solution to (3.25) is contingent on the payoff conditions of the credit risky bond. Now, under the assumptions of constant interest rate  $r$  and deterministic default threshold we can express the time  $t$  price of a credit risky zero coupon bond in terms of the generic formulation:

$$\bar{P}(t, T) = \mathbb{E}_{\mathbb{Q}} \left[ e^{-r(T-t)} \left( FI_{\{\tau > T\}} + \bar{F}I_{\{\tau \leq T\}} \right) \right] \quad (3.26)$$

where  $\mathbb{E}_{\mathbb{Q}}(\cdot)$  denotes the expectation under the probability measure  $\mathbb{Q}$  and  $I_A$  is the indicator function of  $A$ . Additionally, we can express (3.26) in terms of the payoff conditions such that:

$$\bar{P}(t, T) = \mathbb{E}_{\mathbb{Q}} \left[ e^{-r(T-t)} \left( FI_{\{\tau > T\}} + \bar{v}_t I_{\{\tau \leq T\}} \right) \right]$$

where  $F$  is the par value of the bond and is redeemed in the event of no default. Alternately, in the event of default the value of the bond is just equal to the value of the default threshold,  $\bar{F} = \bar{v}_t$ . Following the general formulation for the default probability  $\mathbb{Q}(\tau \leq T)$  from Bielecki and Rutkowski (2002) [14] the price of a defaultable zero coupon bond can be expressed as:

$$\bar{P}(t, T) = e^{-r(T-t)} F \left[ 1 - \mathbb{Q}(\tau \leq T) \right] + \bar{v}_t e^{-r(T-t)} \mathbb{Q}(\tau \leq T)$$

where

$$\mathbb{Q}(\tau \leq T) = N(x_1) + \left( \frac{\bar{v}_t}{V_t} \right)^{2\tilde{a}} N(x_2)$$

$$\begin{aligned}
x_1 &= \frac{\ln \frac{\bar{v}_t}{V_t} - \left(r - \gamma - \frac{1}{2}\sigma_V^2\right)(T - t)}{\sigma_V \sqrt{T - t}} \\
x_2 &= \frac{\ln \frac{\bar{v}_t}{V_t} + \left(r - \gamma - \frac{1}{2}\sigma_V^2\right)(T - t)}{\sigma_V \sqrt{T - t}} \\
\tilde{a} &= \frac{r - \gamma - \frac{1}{2}\sigma_V^2}{\sigma_V^2}
\end{aligned}$$

and  $N(\cdot)$  is the cumulative normal distribution. For ease of implementation we rewrite the credit risky bond price formula as shown in Appendix 3.1.

## 3.8 Credit Spreads

In our study thus far we have shown three different structural models on how to price a credit risky bond. Each theoretical model is distinguished by a set of unique features that extend the seminal Merton (1974) [94] model. While these models are just a subset of a growing literature base it can be said that an important driver of this type of research is the inability of the Merton (1974) [94] model to consistently replicate credit spreads of similar magnitude to those observed in the market. Although some features of each model improve on the limitations of Merton's (1974) [94] model, they still display shortcomings to fully capture market phenomena. In this section we aim to implement each model in so far as to examine the generic patterns predicted by the models, in particular, the structure of credit spreads as determined by leverage and corporate asset volatility. In addition, contrary to literature we do not characterize the models as inadequate to sufficiently predict observed spreads, but rather focus on the determinants that potentially contribute to the shape of market spreads.

The organization of this section is as follows. We start by giving a concise overview of the classic empirical research. Then we define the term structure of credit spreads and explain the rationale behind the parameters that drive credit spreads. In section 3.8.3, we describe the simulation study. In section 3.8.4, we illustrate the simulation results and set forth the objectives for demonstrating the varying term structures of credit spreads. In section 3.8.5, we analyze the results and motivate our findings based on literature. In section 3.8.6, we briefly describe the factors that affect credit spreads. Finally, we conclude with supplementary remarks.

### 3.8.1 Overview of Empirical Research

The pricing of credit risky bonds present some interesting and intractable challenges. While there has been steady progress on the design and improvement of theoretical mod-



els empirical studies to test and validate these models have been sparse up until the past few years. In contrast to the abundance of Treasury bond data good quality corporate bond data was scarcely available. This can be attributed to the fact that traditionally the corporate bond market was illiquid and dealers either had to rely on their models or a database of bond prices to approximate the fair value of an issue that was thinly traded. In addition, it is plausible to assert that the nature of corporate debt structures contributed to the consequence of a limited number of empirical studies. Corporate debt structures are complex and include (i) multiple issues of debt (ii) coupon bearing debt (iii) callable bonds and (iv) sinking fund provisions. Nonetheless, where empirical research was undertaken researchers tended to opt for corporates with simple capital structures. This had the additional consequence of a limited sample size of available risky bonds.

The advances in technology and changes in policy toward propriety information perhaps led to the consolidation and general availability of corporate bond data. A review of the literature shows a small number of empirical studies that attempt to test the efficacy of particular structural models albeit they report results that lack precision relative to observed data sets. In this section we aim to give a concise overview of some of this research to the extent we can relate and motivate these findings to our simulation study on credit spread dynamics.

The first study in this stream of research is widely attributed to Jones, Mason and Rosenfeld (1984) [77]. This research was based on monthly data of 27 firms from the period January 1975 through to January 1981 where available and was restricted to firms with simple capital structures, low leverage and rated debt. The data was tested on a contingent claims analysis model, not dissimilar from Merton's (1974) [94] model, and shows that yields on investment grade bonds were consistently overpriced by on average 9 percent. A similar such study was conducted by Ogden (1987) [100] on 57 new bond issues over the period 1973 to 1985. Here, for the sampling of bonds, credit spreads were underpriced by an average 104 basis points in contrast to market spreads.

The research by Sarig and Warga (1989) [109] analyzes the prices of pure discount bonds of various quality and maturity and compare their term structure of credit spreads to that of similar maturity Treasury bonds. Their data set of monthly zero coupon bond prices were derived from Lehman Brothers data tapes for the period February 1985 through to September 1987 and covers 137 new bond issues across 42 different corporates. They proxy the riskfree yield curve using monthly prices of Treasury strips. The yields on the strips are subtracted from the yields of the zero coupon bonds and by cross-sectionally averaging the yield differentials across all bonds and across time the term structure of credit spreads that qualitatively resemble the Merton (1974) [94] spreads are produced.

Sarig and Warga (1989) [109] suggest that existing theoretical results generally match their empirical observations, more specifically, the term structure of credit spreads is upward sloping for high quality zero coupon bonds, humped-shaped for average quality bonds and downward sloping for low quality bonds. While this sample comprise only zero

coupon bonds that makes for ease of computation it may not be representative of the typical debt structure of a corporate. In addition, since the time-series of data is short this study is ranked as preliminary as compared to suggesting robust empirical evidence. Notwithstanding, this research underscores the practicality of the options pricing framework on characterizing the term structure of credit spreads.

In a comprehensive and recent study Eom et al. (2004) [46] implement a coupon version of the Merton (1974) [94] model together with the Geske (1977) [54], Longstaff and Schwartz (1995) [91], Leland and Toft (1996) [89] and Collin-Dufresne and Goldstein (2001) [26] models in an attempt to evaluate the precision of structural models on pricing credit risky bonds. These models are tested on a sample of 182 non-callable bonds from corporates with simple capital structures. The data was extracted from the Fixed Income Database annually each December for the period 1986 through to 1997. They argue that their results suggest that the five structural models do not give reasonable predictions of market corporate bond prices. In general, they find that some models under-predict credit spreads while other models over-predict credit spreads with spread errors revealing no insight about model mis-specification.

Eom et al. (2004) [46] deduce that most models show low spreads when the associated corporate bonds are from entities with low volatility and low leverage. According to literature low spreads are associated with time to maturity but they find that holding volatility and leverage constant there is no contributing effect from maturity. Of particular interest is the Leland and Toft (1996) [89] model with its simplifying assumptions on coupons it tends to in most cases show an overestimation of spreads. In the main the models by Longstaff and Schwartz (1995) [91], Leland and Toft (1996) [89] and Collin-Dufresne and Goldstein (2001) [26] overcome the problem of low spreads but generally show lack of precision as evidenced by a wide dispersion of predicted spreads. Consequently, this result is amplified if the models incorporate stochastic interest rates and bankruptcy costs as a function of the default recovery rate. They suggest that this could be remedied by incorporating a more realistic term-structure of interest rates. In addition, they argue that the dispersion of predicted spreads is also affected by the valuation of coupons. As a result, they conclude that while fairly stating the risks associated with volatility, leverage and coupon there is scope to improve structural models as to reflect market consistent prediction of credit spreads.

### 3.8.2 Term Structure of Credit Spreads

The credit spread is defined as the difference between the yields of a credit risky bond and its identical risk-less bond. The credit spread is just the risk premium that investors demand for investing in risky debt. The general formula for the spread between the yield on the risky bond of price  $\bar{P}(t, T)$  with maturity  $T$  and par value  $F$  and the risk-less bond  $P(t, T)$  is expressed as:

$$s(t, T) = -\frac{1}{T-t} \ln \frac{\bar{P}(t, T)}{FP(t, T)}$$

The price of the credit risky bond  $\bar{P}(t, T)$  has been derived previously for each model. In addition, for all models we specify the price of a risk-less zero-coupon bond to be expressed as:

$$P(t, T) = e^{-r(T-t)}$$

We rearrange the terms in the credit risky bond pricing formula to show two ratios that have an intuitive economic interpretation (see Appendix 3.1). First, we observe that the parameter in the Merton (1974) [94] model that drives the term structure of credit spreads is  $d$ , generally termed the quasi debt to assets ratio (leverage). Additionally, we define  $r$  as the risk-free rate and  $V$  as the market value of the assets of the firm then the quasi-debt to assets ratio can be expressed as:

$$d = \frac{Fe^{-rT}}{V}$$

Notably,  $d$  does not represent the actual debt to assets ratio since the par value of corporate debt  $F$  is discounted at the risk-less rate  $r$ . Consequently, the discounting factor acts as an upward biased estimate of the true debt to assets ratio. Alternately, the parameter  $d$  can be interpreted as the forward price of assets in the risk-neutral economy. Second, the parameter  $b$  can be described as the bankruptcy or first passage time default ratio. It is simply the ratio of the current value of the default boundary to the current value of the firm. More specifically, when the ratio  $b = 1$  we say the firm is in default.

The term structure of credit spreads  $s(t, T)$  is now completely specified as a function of maturity  $T$ , the firm leverage as measured by the quasi-debt to assets ratio  $d$  and the volatility of the assets  $\sigma_V$ . We will apply these parameters and expressions to compute spreads in (i) the constant default boundary model, (ii) the deterministic default boundary model, and (iii) the stochastic default boundary model.

### 3.8.3 Simulation Study

Although the Merton (1974) [94] model is conceptually elegant its assumptions are somewhat restrictive to allow it to approximate empirical regularities. In this chapter we presented theoretical models that were designed with assumptions to reflect market phenomena and thus reduce potential inefficiencies in structural models. In this section we discuss very briefly the numerical implementation of the three models.

Each of the models has analytical formula for both the risk-free and credit risky bond prices. All the formulas are straight forward to implement and are given in Appendix 3.1. We generate a set of bond prices as predicted by the models and then calculate the corresponding predicted bond yields assuming annual compounding. The term structure

Parameter	Description		Constant	Deterministic	Stochastic
Bond Features	$T$	Maturity			
	$F$	Face Value	750	750	750
	$\kappa$	Bankruptcy Costs			$0.1 P(t, T)$
Firm Characteristics	$V$	Firm Value	300	300	300
			280	280	280
			255	255	255
	$\sigma_V$	Asset Volatility	0.1	0.1	0.1
			0.2	0.2	0.2
			0.3	0.3	0.3
	$\bar{v}$	Default Boundary Level	240	248	250
Interest Rates	$r$	Risk-free rate	0.05	0.05	0.05
	$\rho$	Correlation between $V$ and $r$			0
	$\sigma_r$	Interest rate Volatility			0.2

Table 3.2: Summary of the Model Parameters and Values for the Simulation Study

of credit spreads obtains.

The structural models each have a set of basic parameters that must be estimated. The parameters related to the capital structure and firm value include the initial values of assets and debt, asset return volatility, and those parameters that characterize the default boundary. In addition, implementation of the models requires estimates of parameters that define the risky bond characteristics, as well as parameters related to the default free term structure. Our aim is to simulate the term structure of credit spreads and compare and comment on the trends and patterns of our results relative to empirical findings. As such we do not calibrate our models with estimates of parameters from market data. Instead we choose well-behaved and realistic parameter values to seed our models. In Table 3.2 we show a summary of the key parameters and their values.

The numerical computation is achieved with the following basic parameter values. The firm asset value volatility is set to  $\sigma_V = 0.2$  and the correlation coefficient to  $\rho = 0$ . Both the constant- and deterministic default boundary models are one-factor models while the stochastic default boundary model is a two factor model. We switch off the correlation effects in the stochastic default boundary model by setting  $\rho = 0$  so as to treat this model as a quasi-one factor model. This is because we would like to observe how specific features in each model affect the term structure of credit spreads. The coefficient  $\gamma$  that calibrates

the level of the default boundary is set equal to  $\gamma = 0.2$  for all simulations in the stochastic default boundary model. The bankruptcy costs is set equal to  $\kappa = 0.1P(t, T)$ , a fraction of the value of the risk-free bond price. For the interest rate process we fix the interest rate volatility  $\sigma_r = 0.2$  and the risk-free rate  $r = 0.05$ .

### 3.8.4 Simulation Results

We simulate the term structure of credit spreads for the constant-, deterministic- and stochastic default boundary models. One type of leverage is examined: constant quasi debt to assets ratio,  $d$ . Figures 3.4, 3.5 and 3.6 illustrate the relationship between the level of the credit spread and the time to maturity of the bond for varying degrees ( $d = 2.5$ ,  $d = 2.68$  and  $d = 2.94$ ) of leverage. In addition, for each model, we illustrate the relationship between the level of the credit spread and the time to maturity of the bond at multiples of leverage  $d$ ,  $2d$  and  $10d$ . Figures 5.1-5.3, 5.6-5.8 and 5.12-6.13 (see Appendix 3.1) show the natural consequence of varying the amplitude of leverage. These figures are complemented by Tables 3.3, 3.4 and 3.5 (see Appendix 3.1) where a summary of the values of the credit spreads at each degree and multiple of leverage is displayed.

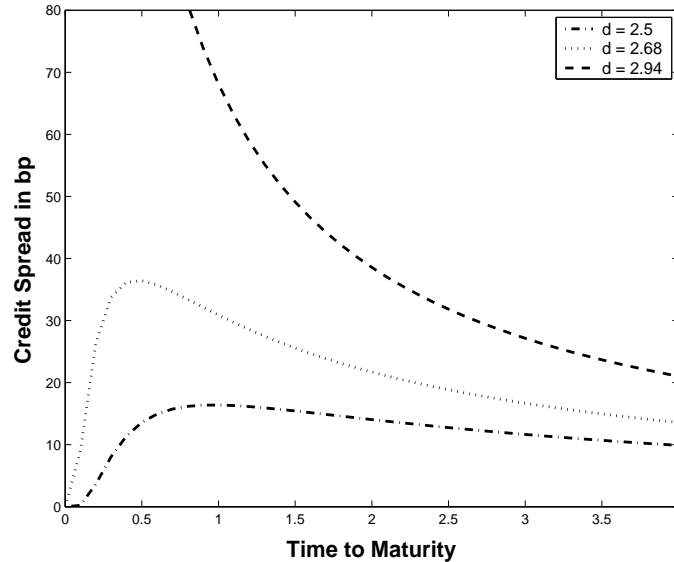


Figure 3.4: Term Structure of Credit Spreads in the Constant Default Boundary Model with varying Firm Leverage  $d$

The main objective of this exercise is to assess the consistency of our results. Additionally, we also want to observe the impact of the default boundary specification on the slope of the curves. In essence we observe the term structure of credit spreads exhibit uniform patterns. For all the models when the degree of leverage is increasing the spread level is an increasing function as the multiple of leverage increases. The patterns of these figures

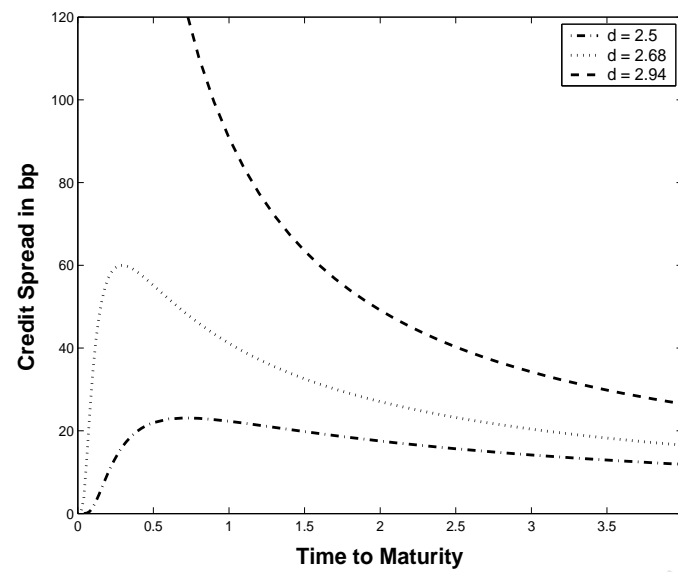


Figure 3.5: Term Structure of Credit Spreads in the Deterministic Default Boundary Model with varying Firm Leverage  $d$

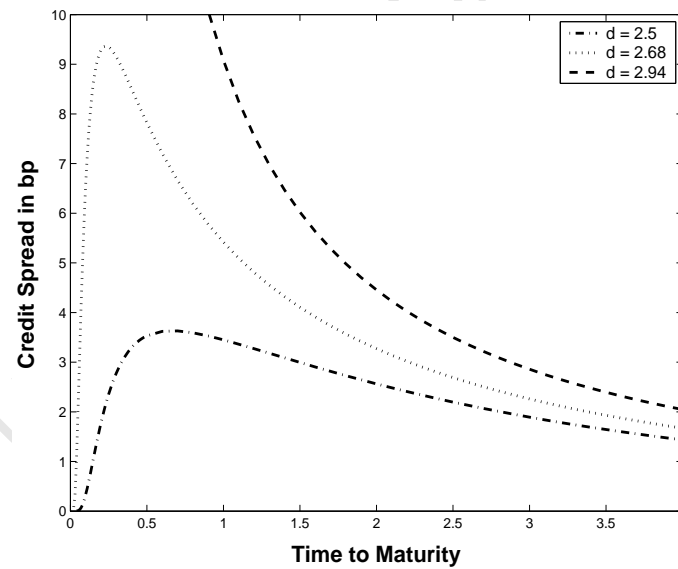


Figure 3.6: Term Structure of Credit Spreads in the Stochastic Default Boundary Model with varying Firm Leverage  $d$

bear an identical resemblance to those produced by Merton (1974) [94]. Characteristically, the term structure is upward sloping for low leveraged corporates, humped shaped for medium leveraged corporates and downward sloping for highly leveraged corporates. The empirical findings by Sarig and Warga (1989) [109] confirm the characteristic term structures shown by the models in this study.

In Figures 5.4-5.5, 5.9-5.11 and 6.14-3.37 (see Appendix 3.1) we capture the effect of corporate asset volatility on the level of credit spreads for the constant-, deterministic and stochastic default boundary models. According to Merton (1974) [94] the credit spread is an increasing function of asset volatility. We simulate two types of asset volatility effects. First, we illustrate the relationship between the level of the credit spread and time to maturity of the bond at varying levels of asset volatility for each degree of leverage. Second, we illustrate the relationship between the level of credit spread and corporate asset volatility at multiples of leverage  $d$ ,  $2d$  and  $10d$  for each degree of leverage. These figures are complemented by Tables 3.3, 3.4 and 3.5 (see Appendix 3.1) where a summary of the values of the credit spreads at each degree of leverage and level of corporate asset volatility is displayed.

The objective of this exercise is to demonstrate the effects of the variation of the asset volatility parameter on the level of credit spreads. For the first effect we observe that the term structure of credit spreads resemble the patterns shown when varying the degrees of leverage. As a result, we say that the term structure is upward sloping for low asset volatility corporates, humped shaped for medium asset volatility corporates and downward sloping for higher asset volatility corporates. For the second effect we observe that the term structure of credit spreads exhibit a single characteristic pattern that resembles the humped shaped curve for medium asset volatility corporates. In addition, this shape is consistent when varying the amplitude of leverage. Intuitively, the consistent pattern displayed by the second effect may suggest the corporate asset volatility parameter is a crucial determinant of credit spreads.

### 3.8.5 Analysis of Simulation Results

The constant- and deterministic default boundary models are one-factor models. The distinctive feature between the constant default boundary model and Merton's (1974) [94] model is that the constant default boundary model is a first passage time model. This realistic feature supposes that the constant default boundary model will produce qualitatively richer spreads relative to Merton's (1974) [94] model. Following this argument we assert that a deterministic default boundary is perhaps more representative of a corporate's debt structure relative to a constant default boundary model and therefore the deterministic default boundary model will demonstrate sharper precision than the constant default boundary model in predicting credit spreads. This is clearly an intuitive deduction. Comparing the numerical results, from Tables 3.3 and 3.4 (see Appendix 3.1), for these two models we observe the credit spreads for the deterministic default boundary model is larger than the constant default boundary model. Notably, this is purely a

preliminary observation based on a simulation study and concurs with the Black and Cox (1976) [18] results.

Similarly, applying this argument to the stochastic default boundary model one may assert that increased randomness in a corporate's debt structure may on average more fairly reflect reality. If this is the case as some researchers believe, for example Briys and de Varenne (1997) [23], then the stochastic default boundary model ought to have better precision than both the constant- and deterministic default boundary models in predicting credit spreads. From Table 3.5 (see Appendix 3.1) we observe that the credit spreads in the stochastic default boundary model is dramatically lower than the spreads that the other two models. First, perhaps this is occasioned as a result of the stochastic default boundary model being a two-factor model. Second, our choice of parameter values could be extreme. Third, only the stochastic default boundary model has a feature that allows for the violation of the absolute priority rule. When such factors are present the spread level is a decreasing function of the quantity of protection afforded by the safety covenant. On the one hand, while this model possess features that reflect real-world phenomena we are tempted to assert that the predicted spreads ought to have a lower margin of error relative to the first two models. Nonetheless, it would be instructive to test the validity of this assertion by calibrating the stochastic default boundary model with market data and evaluating the effectiveness of the model features.

### 3.8.6 Factors that Affect Credit Spreads

A component of the general critique of structural models is that it either under- or over-predicts credit spreads. Several studies such as Sarig and Warga (1989) [109] and Eom et al. (2004) [46], among others, go some way to show evidence that structural models display a degree of mis-specification in predicting credit spreads relative to observed spread levels. While this may be the case, it is also instructive to analyze credit spreads to assess if default risk singularly explains spread levels.

Crouhy et al. (2000) [31] argue that the practical implication of the cross-sectional analysis of credit spreads goes beyond our interest of pricing credit risky bonds and credit risk derivatives and can be extended to the New Basel Capital Accord (2004) [11] requirements for computing a bank's optimal economic capital for credit risk. In addition, Elton et al. (2001) [45] show evidence that only a small component of credit spreads can be explained by expected default loss, for example 17.8 percent for 10 year A-rated industrials. Moreover, they show that the two other factors that account for credit spreads are the tax premium and risk premium for systematic risk. Consequently, this evidence points to a vulnerability of the empirical validity of structural models in explaining spread levels. In the light of these findings new research is being undertaken to decompose credit spreads. In this section we aim to give an overview of the determinants of credit spreads.

The fundamental significance of structural models is that default occurs when the firm value process breaches a specified default boundary. Naturally, these models implicitly



assert that the parameters that drive that firm value process affect the default probability and impact on credit spreads. Avramov et al. (2004) [10] list the determinants that typically drive the credit spread dynamics in structural models as market conditions, leverage, term structure slope, spot rate, firm growth opportunities, stock return momentum and idiosyncratic volatility. In the following we describe the discussion by Avramov et al. (2004) [10] to explain the role of each parameter and its associated effect on the credit spread dynamics.

**Market Conditions:** There is a fair amount of research that explicitly models the relationship between the probability of default and expected recovery rates. For example, the model by Frye (2000) [49] postulates that the default rate is driven by a single systematic factor, that is, the state of the economy. As a result, the same economic conditions that cause an increase in the default rate will cause a decrease in the recovery rate. This shows a negative correlation between the default rate and expected recovery rate. Additionally, credit spreads are negatively correlated with the expected recovery rate. That is, an improving economy implies a higher expected recovery rate and thus lower credit spreads.

**Leverage:** Intuitively, in structural models an increase in the amount of debt implies raising the level of the default boundary closer to the level of the firm value process. From Merton's (1974) [94] model we deduce that an increase in leverage implies a higher default probability with the consequence of larger credit spreads.

**Term Structure Slope:** There are two competing economic interpretations that explain a monotonic increasing term structure of interest rates and its impact on credit spread dynamics. On the one hand, an increasing forward rate curve describes an increase in expected future spot rates that has the immediate implication of decreasing bond prices and hence lower credit spreads. On the other hand, with an increasing term structure slope implying higher expected interest rates impacts on the viability potential of NPV projects that the firm may want to invest in. As a result, investors may place a lower market value on the firm with the consequence of increased spread levels. This paradox of an increasing yield curve slope and its impact on the dynamics of spread level need to be investigated empirically.

**Spot Rate:** In diffusion type structural models the drift parameter of the firm value process is generally positive then conditional on the firm does not default in the short-term then the long-term probability of default is very small. With this foreground Longstaff and Schwartz (1995) [91] show empirical evidence that in a risk-neutral economy an increase in the spot rate results in lower credit spreads.

**Growth Opportunities:** At a corporate level if the forecasts for growth and correspondingly profitability are set to increase then the firm value process will drift away from the default boundary. As a result, the probability of the firm defaulting on its debt diminishes. In addition, Avramov et al. (2004) [10] rely on the expanding literature base

of empirical studies that show evidence of macro-economic conditions and its associated effects on credit spreads. For example, Tang and Yan (2005) [115] demonstrate that macro-economic variables, in particular, risk aversion, current growth rate, and volatility of economic growth, are as important as firm characteristics in explaining a considerable component of credit spreads. Consequently, the firm's growth opportunities is affected by changes in the business cycle. Avramov et al. (2004) [10] proxy market-to-book ratio for future profitability and its variances at both aggregate and firm level to study the effect on credit spreads.

**Stock Return Momentum:** While there has been a considerable amount of research on the cross-section of equity returns that suggest returns are predictable based on historical returns of particular interest is the momentum of stock prices. More precisely, empirical research show that on a periodic moving average basis good performing stocks continue to return better earnings relative to poor performing stocks in the short-to medium term thus producing what is termed momentum in stock prices effect. The empirical implication is that those corporates that bear the higher momentum in equity returns effect tend to be given a higher expected value by investors. As a result, this could imply a farther distance between the firm value process and the default boundary thus returning lower credit spreads.

**Idiosyncratic Volatility:** In their research Tang and Yan (2005) [115] argue that the more volatile a corporate's cashflows are then there is an increased likelihood that it could experience a cash shortfall in meeting its interest obligations. Consequently, the corporate will exhibit a higher probability of default. In addition, it can be asserted that the cost of capital should increase with higher cashflow volatility implying higher credit spreads. Moreover, since the value of the firm is proportional to the current cashflow it can be argued that idiosyncratic equity volatility is proportional to cashflow volatility. On the other hand, Avramov et al. (2004) [10] apply the contingent claim analysis analogy of structural models of a risk-free bond and a short put option on the firm to model credit spread dynamics. More specifically, increased firm volatility makes the option more valuable with a corresponding decrease in the bond prices thereby implying higher spreads. That is, higher firm volatility increase the probability of the firm reaching the default boundary. In particular, they consider the effects of idiosyncratic equity volatility on credit spreads as opposed to total firm volatility that is proxied by market volatility. Both research ideas have their roots in empirical evidence where it was shown that corporate spreads in the 1990's has synchronously moved upwards with idiosyncratic volatility while market volatility has remained constant over that period.

### 3.8.7 Supplementary Remarks

We consider Merton's (1974) [94] model as a benchmark to establish our insight into credit spread dynamics and note two generally accepted deficiencies. First, the model predicts credit spreads that are lower than observed spreads. Second, for short term maturities the model predicts credit spreads that are small or negligible while observed

credit spreads are significant. With this foreground we implement a simulation study of the three structural models that have improved features relative to the standard model. In particular, we test the models at varying degrees and multiples of the two traditional measures of default risk that drive credit spreads, that is, leverage and corporate asset volatility and observe that the patterns of the spreads are consistent with empirical findings. While it is intuitive to assert that increasing randomness of the default boundary is more likely representative of a corporate's debt structure we find that the stochastic default boundary model predicts lower spreads than both the constant- and deterministic default boundary models. Notwithstanding, this is just a preliminary finding and it would be instructive to calibrate the models with market data to evaluate the precision of each model. In addition, Eom et al. (2004) [46] argue that the empirical implication of their evaluation of five structural models is that additional determinants need to be identified to calibrate structural model credit spreads to market levels. We conclude with describing evidence from literature that the firm characteristics may not be the sole determinants of credit spreads and additional potential macro-economic variables contribute to observed spread levels.

### 3.9 Probability of Default

The classical Merton (1974) [94] model postulates that a corporate defaults at bond maturity if assets are not sufficient to pay off the bond obligation. Black and Cox (1976) [18] generalize this model and postulate that a corporate may default the first time the asset value process breaches some lower threshold. Although the original Merton (1974) [94] research did not explicitly set out the modelling of the probability of default subsequent analysis on this work has shown a method to derive the probability of default and it has since become a key measure of credit risk. Eom et al. (2004) [46] show that the prediction precision of the Merton (1974) [94] model for credit spreads is at variance with empirical regularities. While the strong assumptions of the model underlie this phenomenon the empirical implication of this observation is that the model generates default rates that are lower than observed levels. The three models we present in this chapter have modest improvements on the Merton (1974) [94] model. However, the purpose of this section is to show the corporate default dynamics implicit in these models in so far as to observe and compare the term structure of the probability of default to empirical results.

The organization of this section is as follows. We start by giving a concise overview of the classic empirical research. Next, we define the term structure of the probability of default and explain the rationale behind the parameters that drive the probability of default. In section 3.9.3, we describe the simulation study. In section 3.9.4, we illustrate the simulation results and set forth the objectives for demonstrating the varying term structures of default rates. In section 3.9.5, we analyze the results and motivate our findings based on literature. We conclude with supplementary remarks.

### 3.9.1 Overview of Empirical Research

The research by Leland (2004) [88] examines the probability of default generated by the Leland and Toft (1996) [89] endogenous constant default boundary model and Longstaff and Schwartz (1995) [91] exogenous constant default boundary model. When calibrated with base case parameters both models match the characteristic patterns and magnitude of default probabilities reasonably well for terms longer than five years. However, prediction of default probabilities at short-terms tend to be consistently underestimated. The prediction results were compared to the default probabilities observed by Moody's (2001) over the period 1970 to 2000. Additionally, by changing asset volatility only while holding all the other parameters constant and correspondingly making adjustments for leverage across all credit ratings both models can efficiently predict observed longer-term default probabilities. Leland (2004) [88] concludes by suggesting that investigating including jumps in the asset value process may explain the underestimation of default probabilities.

In their study Patel and Pereira (2004) [102] apply prediction-oriented and information-related tests to estimate and analyze the factors that affect the expected default probabilities for a sample of bankrupt and non-bankrupt UK real estate companies. In particular, they empirically test the precision of the Merton (1974) [94], Black and Cox (1976) [18], Longstaff and Schwartz (1995) [91], Leland and Toft (1996) [89], Ericsson and Reneby (1998) and Collin-Dufresne and Goldstein (2001) [26] structural models. Although prediction-oriented tests were biased towards sample characteristics they devise a method to classify errors as type I and type II. Type I error is found to be 5 percent, that is, when the entity defaults and the model fails to predict it. Type II error is found to be 25 percent, that is, when the entity is solvent and the model incorrectly predicts default. In addition, the results show that the estimated expected default probabilities from the different models are closely clustered. In order to validate the accuracy of the prediction results they compute the information related measures of Altman's z-score and a synthetic rating for each entity. It turns out that the information tests show less qualitative results than the structural models. Notably, an important empirical result from this study is that the Ericsson and Reneby (1998) model show generally better performance than the other models in predicting the probability of default.

The empirical evaluation of structural models by Tarashev (2005) [116] focuses on the probability of default component of default risk and examine the general level and time path of the probability of default of each model. The study tests two endogenous default boundary models, Leland and Toft (1996) [89] and Anderson, Sundaresan and Tychon (1996) and three exogenous default boundary models, Longstaff and Schwartz (1995) [91], Collin-Dufresne and Goldstein (2001) [26] and Huang and Huang (2003). Corporate bond data are limited to entities based in the US and are derived from Bloomberg, Datastream and Moody's. An important result from this research is that, in general, model implied probability of default compares well with the time average historical default rates. The best performing model is the Leland and Toft (1996) [89] model but, however, the findings of this research shows marked differences with the study by Leland (2004) [88]. The conclusion of the Leland (2004) [88] paper suggests that the Leland and Toft (1996)

[89] model severely under-estimates the probability of default over the short-term, for example, one-year. The source of the differences in the results can be attributed to the non-linear relationship between the parameter values into the models and the associated sensitivities of the probability of default to these values. In addition, the study finds that the theoretical probability of default does not fully capture the effects of the business and credit cycles on credit risk.

### 3.9.2 Term Structure of the Probability of Default

Recall, in the Merton (1974) [94] model the equity component of the corporate's capital structure can be viewed as a call option on the value of the assets of the corporate with time to maturity equal to the default horizon. Then, based on the model assumptions, if the call option is *in the money* at debt maturity, that is, when the market value of the corporate exceeds the value of its outstanding liabilities, the equity holders will be obliged to redeem the maturing bond obligation. Similarly, if the call option is *out of the money* then the equity holders will let the option expire in default. This conceptual insight of observing the distribution of the payoffs of the call option being out of the money can be theoretically modelled as the distance between the value of the assets of the corporate and the default threshold (bond obligation) or alternately the corporate's default probability or default rate.

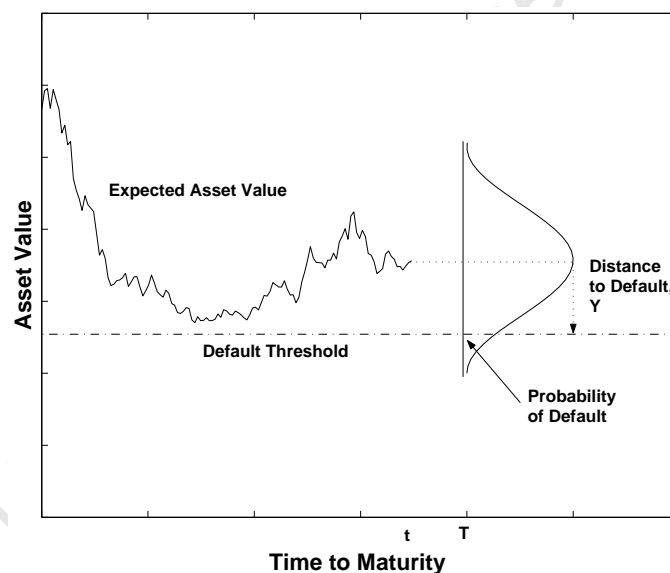


Figure 3.7: Frequency Distribution of Asset Value at Maturity and Probability of Default

A graphical representation of these concepts are shown in Figure 3.7. The vertical axis is assigned the asset value and the horizontal axis shows the time to maturity or default horizon. The total value of the assets is equal to the sum of the market value of

equity and the par value of the bond obligation. At current time  $t$  the market value of the assets is certain and known but the log-normal diffusion asset value process evolves such that an array of asset values is probable at any future date. The array of asset values at maturity date  $T$  is given by its corresponding probability distribution as shown in the figure. In addition, the distribution of the asset values depends on the volatility of the assets. The dashed horizontal line represents the bond obligation due at time  $T$ . If the value of the assets of the corporate at maturity is less than the outstanding debt then the firm will default, *ceterus paribus*. The probability of default is given by the area under the probability distribution below the default threshold and is exactly the probability that the market value of the assets of the firm is less than the value of the debt obligation.

Following Duffie and Singleton (2003) [43] and from this conceptual graphical setting we can formulate the probability of default in mathematical terms. Notice, at current time  $t$  the value of the assets is a random variable. This means that the distance between the asset value and default threshold varies at time  $t$  and we term this the distance to default,  $Y$ . In addition, we define the distance to default to be a geometric Brownian motion with mean  $s = (\mu - \frac{1}{2}\sigma^2)(T - t)$  and variance  $\sigma^2(T - t)$  that can be expressed as:

$$Y_t = \ln V_t - \ln D$$

which is just the number of standard deviations the value of the assets exceeds the value of the debt.

For ease of exposition we first consider the Merton model (1974) [94] with default occurring at maturity  $T$ . The conditional probability of default at time  $t$  is expressed as:

$$P(Y_T \leq 0 | Y_t) = N[x(t, T)]$$

where  $N(\cdot)$  is the cumulative normal distribution function and  $x(t, T)$  is defined as:

$$x(t, T) = \frac{Y_t + s(T - t)}{\sigma\sqrt{T - t}}$$

This is the number of standard deviations by which the distance to default breaches the default threshold at bond maturity  $T$  as seen from current time  $t$ .

Second, we want to generalize this concept of distance to default or probability of default to a first passage time model, that is, the first instant the log-normally distributed asset value process breaches the specified default boundary. Knowing that the probability of survival and the probability of default add up to one, then for each time to maturity  $T$  the probability of survival can be expressed as:

$$P_S(t, T) = P(Y_u \geq 0, t \leq u \leq T) = F(Y_t, T - t)$$

where

$$F(y, u) = N\left(\frac{y + su}{\sqrt{u}}\right) - e^{-2sy} N\left(\frac{-y + su}{\sqrt{u}}\right) \quad (3.27)$$

which is the number of standard deviations by which the distance to default remains above the default threshold. In intuitive terms the first term of expression (3.27) is the standard Merton (1974) [94] formula that represents the probability that the value of the firm's assets would be lower than the value of the default threshold at bond maturity  $T$ . The second term represents the probability that the trajectory of the firm's asset value process would be above the default threshold at debt maturity  $T$  but would breach the default threshold at some time prior to time  $T$ . Expression (3.27) is a generic barrier option formula and can be adapted to compute the cumulative default probabilities for the constant-, deterministic and stochastic default boundary models. The specific formulas for the probability of default for each of the models are shown in Appendix 3.2.

### 3.9.3 Simulation Study

For each model we derive a price for the credit risky bond and a component of the pricing expression is a formula for the probability of default. All the formulas are straightforward to implement and are given in Appendix 3.2. To keep the simulation study consistent with that for credit spreads we retain the basic set of parameters for each of the structural models. The models are not calibrated with estimates of parameters from market data but instead the parameter values were selected because when they were evaluated on the models they were found to be well-behaved and realistic values. Table 3.2, as given in the section on credit spreads, shows a summary of the key parameters and their values. Our aim is to simulate the term structure of default rates (probabilities) and compare and comment on the trends and patterns of our results relative to empirical findings.

The numerical computation is achieved with the following basic parameter values. The firm asset value volatility is set to  $\sigma_V = 0.2$  and the correlation coefficient to  $\rho = 0$ . Both the constant- and deterministic default boundary models are one-factor models while the stochastic default boundary model is a two factor model. We switch off the correlation effects in the stochastic default boundary model by setting  $\rho = 0$  so as to treat this model as a quasi-one factor model. This is because we would like to observe how specific features in each model affect the term structure of credit spreads. The coefficient  $\gamma$  that calibrates the level of the default boundary is set equal to  $\gamma = 0.2$  for all simulations in the stochastic default boundary model. The bankruptcy costs is set equal to  $\kappa = 0.1P(t, T)$ , a fraction of the value of the risk-free bond price. For the interest rate process we fix the interest rate volatility  $\sigma_r = 0.2$  and the risk-free rate  $r = 0.05$ .

### 3.9.4 Simulation Results

We simulate the term structure of default rates for the constant-, deterministic- and stochastic default boundary models. One type of leverage is examined: constant quasi debt to assets ratio,  $d$ . Figures 3.8, 3.9 and 3.10 illustrate the relationship between the level of the default rates and the time to maturity of the bond for varying degrees ( $d = 2.5$ ,  $d = 2.68$  and  $d = 2.94$ ) of leverage.

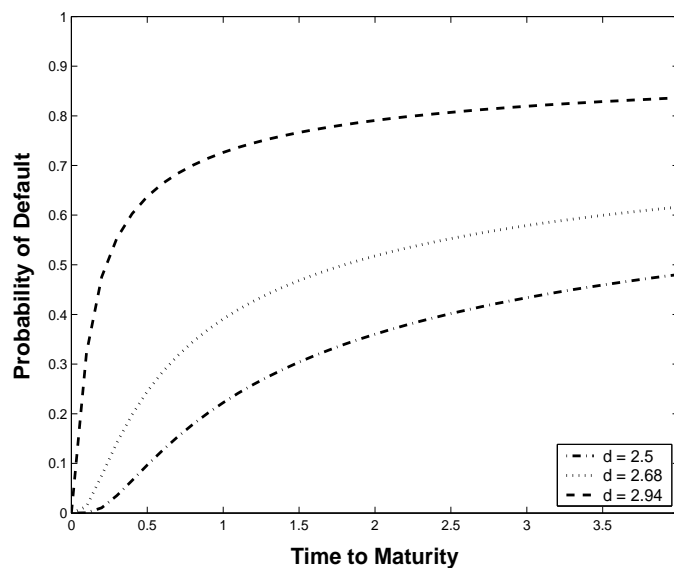


Figure 3.8: Term Structure of the Probability of Default in the Constant Default Boundary Model with varying Firm Leverage  $d$

The main objective of this exercise is to assess the consistency of our results. Additionally, we also want to observe the impact of the default boundary specification on the shape of the curves. In essence we observe the term structure of default rates to exhibit uniform patterns. Unlike the simulation results for credit spreads the relationship between the level of the default rate and the time to maturity of the bond is not affected at multiples of leverage  $d$ ,  $2d$  and  $10d$ . Characteristically, the term structure is upward sloping for low-, medium- and highly leveraged corporates. Moreover, the term structure for highly leveraged corporates shows an initial sharp incline followed by a gradual flat structure.

In Figures 3.38-3.41, 3.42-3.45 and 3.46-3.49 (see Appendix 3.2) we capture the effect of corporate asset volatility on the level of default rates for the constant-, deterministic- and stochastic default boundary models. We simulate two types of asset volatility effects. First, we illustrate the relationship between the level of the default rate and time to maturity of the bond at varying levels of asset volatility for each degree of leverage. Second, we illustrate the relationship between the level of the default rate and corporate asset volatility at varying degrees of leverage.

The objective of this exercise is to demonstrate the effects of the variation of the asset volatility parameter on the level of default rates. For the first effect we observe that the term structure of default rates resemble the patterns shown when varying the degrees of leverage. As a result, we say that the term structure is upward sloping for low-, medium-



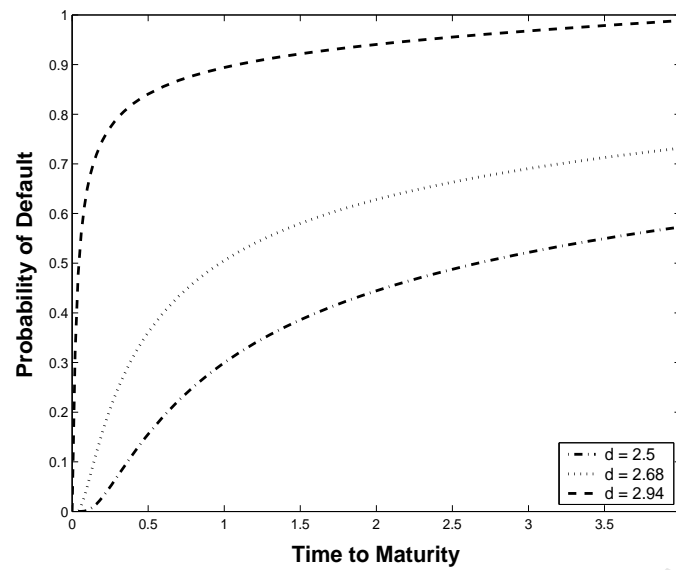


Figure 3.9: Term Structure of the Probability of Default in the Deterministic Default Boundary Model with varying Firm Leverage  $d$

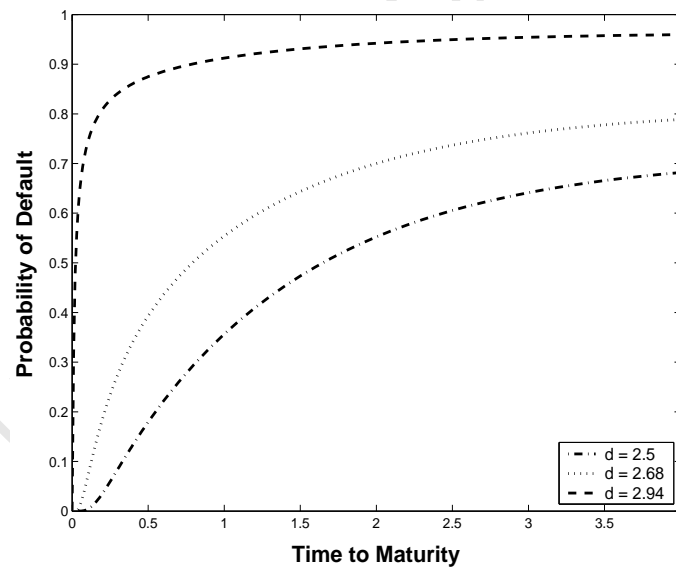


Figure 3.10: Term Structure of the Probability of Default in the Stochastic Default Boundary Model with varying Firm Leverage  $d$

and high asset volatility corporates. Notwithstanding, for low corporate asset volatility both the constant- and deterministic default boundary models show a low upward sloping gradient. For the second effect we observe that the term structure of default rates exhibit a single characteristic pattern that resembles an initial sharp incline followed by a gradual linear structure as evidenced for high leverage ( $d = 2.94$ ) corporates. In addition, a peculiar phenomenon occurs for the stochastic default boundary model where for each level of volatility all the curves increase sharply and closely together and converge at approximate time to maturity of 4 years and trends linearly thereafter. This effect could be attributed to the stochastic interest rate assumption of the model in conjunction with the neutral correlation between interest rates and implicitly the corporate asset volatility. Intuitively, the consistent pattern displayed by the second effect may suggest the corporate asset volatility parameter is a crucial determinant of default rates.

### 3.9.5 Analysis of Simulation Results

On the analysis of structural models we posit that the assets to debt ratio or leverage is a predictor of credit risk. However intuitive this may appear leverage, on the contrary, has low predictive power for credit risk. Kealhofer (2003) [81] argues that generally, investors are risk averse and will thus invest in the safest projects or the projects that have the lowest risk of failure. This fact can be seen by increasing the multiples of leverage the default rates remain unaffected. In addition, as evidenced in the figures is the cross-sectional relationship between leverage and corporate asset volatility, that is, with increasing leverage corporate asset volatility trends lower. The empirical implication is that corporates with lower asset volatility use more leverage. Generally, for all models the probability of default increases with time to maturity for low leveraged corporates and sharply increases and then trends linear for higher leveraged corporates. Nonetheless, all levered corporates show an increasing trend in the probability of default. As a result, any corporate with some debt will default the longer the horizon to maturity.

Ideally, the credit risk models in this chapter have to be calibrated with market data to discern a prognosis on their default probability precision. Notwithstanding, the trends and the shapes for the varying degrees of leverage and levels of corporate asset volatility are generally consistent with empirical literature, see for example Leland (2004) [88].

### 3.9.6 Supplementary Remarks

In this section we focus on the stream of analysis that goes beyond the original Merton (1974) [94] research and simulate the model implied probability of default for the constant-, deterministic- and stochastic default boundary models. While, traditionally credit spreads have been used as a credit risk measure of an obligor studies in literature, for example Avramov et al. (2004) [10], have asserted that several other factors including credit risk affect credit spreads. A purer measure of credit risk is implied by the probability of default. On the other hand, the market price of a corporate bond reflects its

time value, liquidity premium, value of embedded options, credit risk and the like. In each model we show a closed-form formula to extract the probability of default from the price of the credit risky bond. We test the models at varying degrees of the two traditional measures of credit risk, leverage and corporate asset volatility, and observe that the patterns of the term structure of the probability of default is consistent with empirical findings. While it is intuitive to assert that a larger leverage ratio may increase the probability of a corporate to default the models correctly show that increasing leverage has no effect on the probability of default. In addition, the volatility of the probability of default ought to show some correlation with the corporate asset volatility since the more volatile the assets of the corporate the increased likelihood of default risk. As an effort for future research it is instructive to quantify and analyze the effects of default probability volatility. The above simulation exercise is an encouraging exercise towards gaining insight into the empirical performance of the term structure of the probability of default.

### 3.10 A Strategic Analysis of Structural Models

The research initiated by Merton (1974) [94] was the first attempt to generalize the option pricing theory of Black-Scholes (1973) [20] towards a theoretical framework on structural models of pricing credit risky bonds. Consequently, this work has alternately become known as the contingent claim analysis and an important example of which is the pricing of the equity and debt components of a corporate's capital structure as contingent claims. While Merton's (1974) [94] model is an elegant mathematical formulation for pricing credit risky bonds it, however, makes several simplifications to derive the final pricing formula. For example, one such simplification is the corporate's capital structure. A typical capital structure consists of equity and multiple issues of debt. This is in contrast to debt in Merton's (1974) [94] model where debt is represented as a single issue zero coupon bond. Of particular importance is this characteristic of complex debt structures and its impact on the pricing of credit risk. Second, a potential factor that explains the apparent inefficiencies of structural models to fairly predict the probability default and credit spreads especially at times close to maturity could well be attributed to the arbitrary and simplified assumptions of the default boundary. Third, the procedure of bankruptcy represents a statutory domain for contract renegotiation when various stakeholders of the firm cannot reach agreement following a default on the debt obligation. Importantly, the bankruptcy process has scope for the violation of the absolute priority rule. As part of the strategic analysis of structural models we aim to assess the effect of interactions of capital structure, cross-sectional analysis of the default boundary and the bankruptcy process and its concomitant results on the pricing of credit risk.

### 3.10.1 Capital Structure

The search for the optimal design of the debt contract has long been a challenge in the theory of capital structure. Commonly, individual firms endogenously assess their level of risk and this forms the basis of their optimal debt level. In this chapter we show models with the capital structure identical to that shown in Merton's (1974) [94] firm value model. More specifically, we choose a vanilla capital structure of equity and a single zero coupon bond. Additionally, the capital structure is static throughout the tenure of the debt obligation. In this setting credit risk is easily identified and priced but, however, these models are fundamentally basic to typical capital structures. Moreover, these contingent claim default models belie the traditional issues in corporate finance that help explain the risks associated with default. By considering complex capital structures we can identify the factors determining the firm's financing decisions thereby guiding our decisions in the design of the optimal debt contract.

Modigliani and Miller (1958) [95] are widely accredited with the seminal work on the modern theory of capital structure where they show through no arbitrage arguments, and based on a set of assumptions, that capital structure is irrelevant. Several improvements followed this research, for example, agency theory by Jensen and Meckling (1976) [75] and information asymmetry by Myers and Majluf (1984) [98], emphasizing particular features in models to illustrate optimal debt design. With this foreground research we formulate our analysis along the two thrusts, the cost of agency and information asymmetry, respectively, to motivate that investors recognize that default is a strategic decision taken by the firm. In this section we give the empirical implications as put forth by the interaction between credit risk, risk premiums and the firm's capital investment decisions.

#### 3.10.1.1 Costs of Agency

In their paper Harris and Raviv (1991) [60] present a well researched survey on the theory of capital structure. They point to an expanding base of research on the costs of agency that yield crucial insights on capital structure dynamics. We rely on the virtues of this study to exposit the conjunction of capital structure and the pricing of credit risk.

The traditional research on the costs of agency and its effects on capital structure is widely attributed to Jensen and Meckling (1976) [75]. The type of conflict they identify, and one that concerns us, is that between equity holders and debt holders. This arises because the debt contract is designed such that equity holders can maximize gain for themselves. In particular, the debt covenants stipulate that if a project yields returns in excess of the debt outlay then the equity holders benefit this excess. In addition, if the project yields losses and because equity holders possess the limited liability right debt holders bear these losses. Moreover, investing in the risky project will result in a loss in the value of equity but the gain in equity value extracted from the debt holders more than compensates for this loss. Consequently, equity holders stand to benefit from potentially loss making projects while the value of debt is seen to decrease. Conversely, suppose

if debt holders correctly predict the equity holders investment behavior then the risk of the yields on the risky project is borne by the equity holders. More specifically, equity holders can only dispose of the debt at an additional discount to the price the market would usually offer. As a result, equity holders who issue debt to invest in risky projects bear this cost to debt holders. This describes the agency cost of debt financing and is commonly termed the *asset substitution* effect.

Following Harris and Raviv (1991) [60] we list several remedies to reduce or eliminate the agency cost of debt to effect an optimal capital structure design. First, bond contracts should be designed to embed clauses to discourage asset substitution, such as coupon payment provisions or exclusion of investments in new, non-core business projects. Second, there is an apparent positive correlation between industries with few opportunities for asset substitution and higher debt levels. Third, if firms forecast their optimal growth to be weak or even negative while in the same period they have large revenue inflows from operations then these entities should have increased debt. Clearly, the overwhelming implication of the costs of agency between equity holders and debt holders is that while increasing debt disrupts manager's tendency to consume cashflows for his personal benefit it also captures some of the retained cash and increases the manager's proportion of ownership of the residual claim.

The theory of the costs of agency also encompasses the branding of managers or firms conscientiousness of their reputations with regard to investment decisions. Included, in their survey Harris and Raviv (1991) [60] review two studies that suggest that the inherent consequences of reputational effects lead firms and managers to invest in relatively safe projects. This factor reputedly strives to mitigate the agency cost of debt.

The first study models the reputational effects of a firm investing in projects that generate sufficient cashflows to repay the invested debt. If a firm pursues a short term strategy of maximizing profits, then conditional on the asset substitution effect, the firm will choose the risky investment. The firm will enjoy a lower interest rate if it can convince lenders that the project has a positive net present value (NPV). Note, the firm's history is in the public domain and it can enhance its default free reputation by successively choosing safe projects. A long default free history underpins a good firm reputation and attracts cheaper interest rates. Firms with a good reputation tend to be older, have a long history and will act to protect a good reputation. They will not engage in asset substitution and will choose to invest in safe projects. On the other hand, young firms with a fairly unknown reputation may want to maximize their short-run profit outlook and opt to invest in a risky project. If there is no default during the lifetime of the project they will eventually invest in the safe project. The implication of this study is that firms with a good reputation have long histories and tend to have lower default rates and cheaper interest rates, thus lower agency costs of debt, than their relatively younger counterparts.

The second study concerns managers who serve to protect their reputations and will therefore choose to pursue safe investments. From the manager's point of view the labour

market can only distinguish between *success* and *failure* on a project. On the other hand, from the equity holders perspective a risky investment promises higher expected returns and if successful higher returns. As a result, the manager maximizes the likelihood of success while equity holders prefer higher expected returns. Conditional on preserving a good reputation the manager will always choose a safe project that has higher expected returns for equity holders. Consequently, the manager's behavior reduces the agency cost of debt. Thus if managers are concerned about their reputations then it is likely that the firm will carry more debt than usual.

We derive utility from agency models to the extent of its versatility to show the distribution of factors that affect the pricing of credit risk. More specifically, these models show that the leverage is positively correlated to firm value, default probability and the reputation of firms and managers. On the other hand, these models predict that leverage is negatively correlated with the measure of growth prospects and borrowing costs.

### 3.10.1.2 Information Asymmetry

The research by Myers and Majluf (1984) [98] sets out the case that information asymmetry is an additional determinant of explaining capital structure. In this model it is assumed managers have more knowledge of the firm's opportunities, risks and cash-flows than investors. Clearly, managers do have more information about the firm than investors. We can test this assertion by observing the share price movements that track the announcements of managers. Generally, when a firm issues an increased dividend, the share price typically increases and investors interpret this increase as a signal of management's confidence in the future cashflows of the firm. Implicit in this action is that information is being transferred from managers to investors. This can only occur if managers have information asymmetry. In this section we concern ourselves with the stream of literature where capital structure is designed to compensate for the effects of information asymmetry and thus allow firms to efficiently evaluate investment projects.

Myers and Majluf (1984) [98] show in their model that investors with imperfect information, relative to that of the firm's managers, with respect to the value of the assets of the firm and its opportunities can result in the mis-pricing of equity. In some cases firms need to issue equity to invest in new projects. This equity could be substantially undervalued such that new investors can capture in excess of their fair proportion of the NPV of the new project and therefore to the detriment of old equity holders. These positive NPV projects dilute the holdings of old equity holders and are summarily rejected. The remedy for this under investment paradox is to fund the new project with a security that is fairly priced by the market. The firm's cash reserves or its ability to issue risk free debt cannot be mispriced by the market and as such will be preferred to an issue of new equity to finance a new project. The investment decision based on this selective preference over the different components of the capital structure is commonly known as the *pecking order* theory in corporate finance literature.

We describe the empirical predictions of the Myers and Majluf (1984) [98] study of information asymmetry on capital structure. First, if the firm announces to the market its intention to issue new equity to finance a new project this will result in a decrease in its present share price. Second, where possible the firm will try to avoid this scenario and will attempt to finance a new project with cash reserves or with a security whose price is not or barely sensitive to private information. Finally, consider a firm with a low ratio of tangible assets to firm value then such a firm is more susceptible to information asymmetries. The market will frequently underprice the securities of these firms relative to firms with marginally better information asymmetries. Consequently, these firms tend to accumulate more debt in the long term, *ceterus paribus*.

The tenure of the debt issue under asymmetric information is a crucial determinant in the arbitrage free pricing of the debt obligation. In their research Goswami et al. (1995) [57] examine the effects of the relative distribution of asymmetric information with respect to short and long-term cashflows on the design of debt obligations.

The analysis of their results demonstrate several important empirical implications. If it is deemed that the density of the information asymmetry is forecast for periods in the long-run then the firm will prefer to issue a long-term indentured bond obligation. On the other hand, if the firm predicts greater informational asymmetry in the near term then the more likely mode of finance will be a short term debt obligation. The firm will likely opt for indenture free long-term debt if informational asymmetry is deemed to be prevalent in the short term and less likely concerning long run income. Moreover, the compliance with corporate regulation through the periodic release of accounting reports serves to give a more predictive outlook on the short-term cash flows and thus tend to reduce the information asymmetry in the near term. This analysis show the effects of the arrival of informational asymmetry on the risk of default.

### 3.10.1.3 Supplementary Remarks

Admittedly, the costs of agency and information asymmetry are intuitive but are conspicuously absent in the models we present in this chapter. The explanation for this practice is that the basis for these features are qualitative and not mathematical constructs. This makes it difficult for these features to be explicitly calibrated into formal models and as a result its effects cannot be quantified, no empirical analysis can be validated to guide researchers to differentiate between models to seek out the optimal debt contract. Nonetheless, a survey of literature indicates that some researchers are attempting to reflect these qualitative features in structural models. These models have the characteristic of the endogenous default decision and one of the earliest such models is by Black and Cox (1976) [18] with a dynamic theory of capital structure.

We observe that the pricing of credit risky bonds is linked to the choice of its capital structure. More precisely, in structural models the contingent claims valuation of credit risky bonds can only be priced if the capital structure of the firm is known. On the

other hand, we can only design the optimal capital structure if we can fairly assess the risk of leverage on bond prices. It thus remains that structural models for pricing credit risky bonds should incorporate a quantitative construct for an optimal theory of capital structure, by modelling the dynamics of the price of credit risk and capital structure analytically.

### 3.10.2 The Default Boundary

An underlying characteristic of structural models is the economic basis of the default decision that in turn is fundamental to the pricing of credit risk. In the credit risk models we present we explicitly specify the default condition. Notwithstanding that the default boundary is an important building block in structural models existing assumptions with regard to the default criteria have rarely been analyzed and validated. Clearly, if default is triggered by idiosyncratic factors for different firms then the general models will lack precision in their predictions. It remains that unless our insights into the determinants of the default boundary improves considerably then our ability to fairly price credit risky bonds is likely to be limited. The main objective of this section is to present an analysis of the default boundary to gain insights into the design of a realistic default boundary that is consistent with empirical observations.

In this chapter we present three analytical valuation models for credit risky bond prices and the main feature that distinguishes each model is the default boundary. Following Merton's (1974) [94] model we have a constant default boundary model with constant interest rates but includes the potential outcome of default before the bond's maturity. The second model is similar to the Black and Cox (1976) [18] model where the default boundary is a deterministic function of time. The third model is the stochastic default boundary model of Schönbucher (2000) [112] where he extends the credit risky bond pricing model of Black and Cox (1976) [18] to allow interest rates to follow the Vasicek (1977) [118] diffusion process. In addition, Schönbucher (2000) [112] defines the default boundary as a fixed quantity discounted at the risk-free interest rate up to the maturity date of the credit risky bond. This results in the model being characterized by a default boundary with the stochastic dynamics of the risk-free interest rate.

In both the constant and deterministic default boundary models we have a single bond obligation with no new debt being issued during the tenure of the bond. The models therefore predicts that the expected leverage ratio will decrease exponentially over the horizon of the bond's time to maturity. This decline in expected leverage ratio is, however, not validated by empirical regularities. In contrast, the Schönbucher (2000) [112] model specifies the default boundary is co-integrated with the dynamics of the risk-free interest rate over the horizon of the bond's time to maturity. As a result the default boundary evolves proportionally with the firm value over time and the expected level of leverage thus remains constant. This is a plausible assumption in the event the firm cannot alter its expected level of leverage over time.

All the models specify the default boundaries to follow rigid dynamics and as such



are restricted to these modes with no mechanism to incorporate other default cases. For example, a typical scenario of default is when the firm faces a liquidity crisis to repay short-term debt even though the assets of the firm is above the default boundary, that is, the total outstanding debt obligation. In this case the default probability is high and the default boundary in each of the models cannot anticipate this scenario.

On structural models of pricing credit risky bonds the default boundary represents either a safety covenant, which is essentially a contractual agreement that gives the bondholders the right to reorganize the firm when its asset value crosses a defined level or a default trigger deduced mainly by the firm's current liabilities. Generally, a credit risky bond has a long-term horizon to maturity which implies that its default boundary is likely to vary with time. We follow the research by Davydenko (2005) [35] to identify and describe different empirical variables that can be used to proxy model parameters that determine the default decision.

**The Market Value of Assets:** We denote the firm value to be equal to the value of its assets. The continuous market value of the assets can be proxied by the sum of the total market value of the equity and the total market value of the debt. The value of the debt is derived by observing monthly bond prices. The market value of the bank debt can be approximated by applying the contemporary index yield spread between high yield loans and bonds.

**Costs of External Financing:** Generally, banks are the usual source of external financing to distressed corporates. Financing may more likely be extended to corporates with high net value assets that are not already bonded as collateral. The high net value assets can be estimated as the sum of property, plant and equipment plus current assets. Thus a proxy for borrowing costs is the nominally authorized but undrawn lines of credit which is one minus the borrowed debt divided by the total authorized credit limit. The costs incurred to raise the loan finance is measured by the financing costs of the last loan agreement provided this was concluded at most in the past two years.

**Liquidation Value of Assets:** The value of the firm's assets at bankruptcy is contingent on their useful life and tangibility. The market value of such assets should also be noted. The general proxy is tangible assets, that is, plant, property and equipment, and current assets. This is a measure of high net value assets and are usually standard firm assets, and therefore do not markedly lose value at default. Intangible assets is an associated proxy. On the other hand, general industry conditions need to be observed. For example, if the entire sector is in distress then this could impact on the liquidation value of assets.

**Debt Wealth Transfers:** The current value of debt plus the market value of long-term debt minus the recovery value at default is just the discounted value of debt. Accordingly, increased asset volatility reduces market value of long-term debt and increases the value of the call option in the hands of the equity holders. The primary proxy used is the median equity volatility for firms in the same sector.

**Equity Holders' Bargaining Power:** It is commonplace in corporates for managers to be equity holders and as such serves as a prime motivating incentive for managers to assert a tough stance in renegotiations over other claimants. As a result *managerial shareholding* serves as an ideal proxy for equity holders' bargaining power and is measured as the ratio of the number of ordinary shares owned by the five highest paid executives to the total number of shares outstanding. *Institutional shareholding* which is the total equity held by asset managers, pension funds and the like is also a useful proxy. Consequently, sophisticated shareholders can form coalitions and negotiate to capture excess gain at the expense of other less organized shareholders.

**Liquidity Position:** Generic accounting measures are employed to assess the liquidity status of the firm. A common proxy used is the *quick ratio* which is just the sum of cash and near cash plus accounts receivable divided by the current liabilities. This ratio is negatively correlated with liquidity flows the firm maybe encountering.

### 3.10.2.1 Supplementary Remarks

In his study of the default boundary Davydenko (2005) [35] suggests that the assumption of default as triggered by the asset value of the firm breaching some threshold is not consistent with the empirical findings. One plausible explanation of this occurrence is that the default boundary is not compositely calibrated to the relevant firm factors that influences default risk. On the other hand, some firms with low asset values do not default which suggests that the default boundary modelled on observable firm variables may not be the sole predictor of default. Consequently, this mixed outcome of the default boundary modelling is likely to encumber the efficiency of structural models to predict the probability of default and credit spreads.

In the following we describe the recommendations that Davydenko (2005) [35] cites to overcome the default boundary limitation on structural credit risk models. The default boundary should be robustly empirically modelled such that its value can be sufficiently explained by an array of explanatory variables. A more radical approach is to treat either the default boundary or asset value of the firm as unobservable. Some researchers, for example, Duffie and Lando (1997) [40] calibrate information asymmetry into structural models and show that such settings are similar to reduced form models where the default event is unpredictable. Based on the evidence of the default boundary study Davydenko (2005) [35] suggests that a possible approach to advance credit risk modelling is to accept a degree of randomness in structural models.

### 3.10.3 Default and Bankruptcy

In structural models of credit risk the modelling of the default probability is crucial in valuing the equity and debt components of a firm's capital structure since it determines the value of the firm and its apportionment to the various stake holders upon default. Of

the models we present in this chapter we apply two key valuation insights of the Merton (1974) [94] model. First, the lower default boundary is specified exogenously. Second, in the event of default the value of the firm that bond holders receive is known at the outset. Further, Black and Cox (1976) [18] was the first to point out that equity holders bear the limited liability right and they can implicitly trigger default. The underlying economic interpretation of this foreground suggests that the equity holders' decision to default depends on the value that they can capture from the firm in conjunction with the level of protection it possesses but, however, cognisant of the associated costs of financial distress. In this section we show the limitations of these assumptions and the impact of bankruptcy on the decision to default.

To get an in depth insight into the nature of structural models let us consider, for example, the analysis of the constant default boundary model for the zero coupon credit risky bond,  $\bar{P}(V, t, T)$ , that imposes the default conditions of (i)  $\min\{F, V\}$  if  $t = T$  or (ii)  $V$  if  $t < T$ . This says that the bond value at maturity equals the minimum of either the par value  $F$  of the bond or the firm value  $V$ , or the bond value at default equals the value of the firm  $V$ . The economic feature represented by this model boundary condition is the bankruptcy process. Similarly, the deterministic default boundary in this study also assumes that upon default of the bond obligation the bond holders use the bankruptcy procedure to seamlessly and without cost to take over the assets of the firm to recoup their investment. Notwithstanding, this conservative assumption is generally implicit in most models of contingent claim analysis. As a consequence, it has an important impact on the performance of structural models.

In our study both the constant and the deterministic default boundary models are consistent with Chapter 7 of the U.S. Bankruptcy Code where default coincides with the liquidation of the firm's assets. In contrast the stochastic default boundary model assumes that at default the value of the firm less a specified bankruptcy cost is available to the claim holders. The bankruptcy feature of this model is consistent with Chapter 11 of the U.S. Bankruptcy Code where firms under financial distress can apply for protection under this procedure to prevent bond holders from capturing firm value by liquidating the firm's assets.

The occurrence of the bankruptcy process has not gone unnoticed by researchers and a sizeable amount of the inquiry has been initiated to analyze the consequences of bankruptcy law and its effects on the ongoing operations of the firm. Two such empirical studies by Weiss (1990) [119] and Franks and Torous (1994) [48], respectively, report that (i) for a firm to be placed in the state of bankruptcy is expensive both because of its direct associated costs and its unquantifiable opportunity costs (ii) the various stakeholders of the firm exploit the bankruptcy procedures for maximum gain and (iii) priority of claims are generally violated during bankruptcy proceedings. Taken together these factors that emerge from the firm bankruptcy process represent a potential determinant of mispricing of credit risky bonds. Of the three models we present just the stochastic default boundary model incorporates the feature of bankruptcy costs that accords it a realistic state vari-

able when performing analysis on financial distress. Bankruptcy in the stochastic default boundary model is determined by a deviation of the absolute priority rule and is limited to calculating its costs as a fraction of the risk free bond but the model does not, however, reflect the empirical facts of (i) and (ii) above. As a consequence, this model may require implausibly high values of bankruptcy costs or firm volatility to generate credit spreads that match those observed in the market. In the discussion that follows we give a concise discussion of the three stylized facts identified from literature.

**Bankruptcy Costs:** Investors have long been aware that the potential costs of bankruptcy is a single element in an array of risks that need to be priced when valuing corporate debt. The costs of bankruptcy can be both direct and indirect. Direct costs are quantifiable and include legal and professional services and its support and administrative costs related to the Chapter 11 filing. On the other hand, the indirect costs of bankruptcy are unquantifiable opportunity costs that can result from (i) flight to safety by clients and its associated decline in revenues and inventory (ii) employees and suppliers may demand a premium to maintain their relationship with the firm that implies increased operating costs and (iii) with the executive and management distracted by the bankruptcy proceedings competitive tendencies of the firm may be inhibited and can result in the firm be viewed as a potential takeover target. In the study by Weiss (1990) [119] he finds direct costs to be on average 3.1% of the book value of debt plus the market value equity of the financial year-end values preceding bankruptcy. The direct costs of bankruptcy in this study appear to be lower than the previous studies and can be attributed to the characteristics of the sample data, that is, the size and the type of the firms studied, methodology of the study and the evolution of bankruptcy law. Weiss (1990) [119] suggests that this evidence shows there will be little or no impact on the pricing of credit risky bonds.

Bond holders are aware that bankruptcy is costly and they bear the costs of bankruptcy such that they tenuously accept the notion of strategic default, that is, they anticipate some likelihood that the firm will not fully meet its contractual obligation. Typically strategic default is when the firm value falls below an implicit trigger that is determined by the equity holders. As a result, in a rational expectations model bond holders correctly impute the cost of the equity holders gain in the price of credit risky bonds.

**Bankruptcy Procedures:** The legal authority of bankruptcy procedures is inclusive to the extent that it confers rights to unsecured creditors and equity holders, rights these junior claimants would not hold outside bankruptcy. Sophisticated junior claimants will be quick to identify opportunities and protract proceedings to capture wealth from the original claimants. Equity holders can resolve to influence the restructuring process of the firm through management but, however, once in financial distress the usual incentives they have on offer to orientate management toward their objectives is not available. As a result this can cause tension in the agency relationship between equity holders and management. On the other hand, Betker (1995) [12] argues that while equity holders influence over management is weakened under Chapter 11 proceedings creditors appear to gain on opportunities to influence management. For example, empirical evidence suggest that at

the demand of creditors executives and management are often replaced. Alternately, when managers are allowed to continue with the firm some common initiatives that creditors use to gain management's co-operation include threats to alter compensation contracts and vetoing reorganization plans. Consequently, bankruptcy procedures affect the relationship between management, equity holders and creditors and the direct apportioning of firm value.

**Absolute Priority Deviation:** Impaired claim holders maybe willing to accept a deviation from absolute priority in order to receive a timely payment of their share of the firm value. Franks and Torous (1994) [48] offer an appealing explanation of equity deviation. They suggest that there exists a quid pro quo between equity holders and creditors, that is, deviations from absolute priority imply creditors receive reduced payments while equity holders offer not to exercise their option to delay payment to creditors. The option to delay is equivalent to issuing a threat to file for Chapter 11 proceedings or delaying the firm's restructuring process. A period of delay can result in the value of the firm's assets gaining in value over the firm's debt but, however, delay can also increase the costs of default that creditors bear. Consequently, deviations from absolute priority is treated as the time value of the option. It measures the additional discount on the firm value the creditors will give up relative to what they would if the absolute priority rule is observed. The empirical implication of the option to delay is that the time value of the option increases as the option gets closer to the money or, more specifically, the value of the firm gets closer to the value of the creditors claims. As a result equity holders can extract a larger absolute priority deviation from creditors.

In some cases of financial distress creditors initiate a management change to the effect that the new management is creditor friendly. As a consequence, equity holders bargaining power is diluted and this can result in smaller absolute priority deviations. In addition, Weiss (1990) [119] shows evidence of a positive correlation between equity deviation and size of the distressed firm where size is assumed to reflect diverse claimants. Generally, a large firm size will have larger numbers of investors that in turn stake their claim on the firm value. This can imply less formal organization among claimants that can result in relative higher equity deviations. On the other hand, the more claims held by sophisticated and institutional investors can result in better renegotiation agreements for these investors with tighter deviations from priority. Clearly, these empirical proxies motivate that deviations in absolute priority are common in Chapter 11 workouts.

### 3.10.3.1 Supplementary Remarks

While there has been considerable success in developing theoretical models for the pricing of treasury bonds, options and other derivative contracts, the parallel development of the contingent claims approach to risky debt valuation has consistently failed to replicate observed market prices. For example, the comprehensive empirical analysis on the three structural models conducted by Eom et al. (2004) [46] shows that these models experience difficulty in predicting credit spreads. Some models underpredict spreads

and others overpredict spreads. Eom et al. (2004) [46] conclude that the challenge is to reconcile the theoretical bond pricing models with the observed spreads all the while incorporating fair estimates of volatility, leverage and coupon. It remains that credit risky bond pricing models need to be formulated to include these strategic features of default and bankruptcy in order to consolidate their basis as an image of modern corporate finance theory.

## 3.11 Conclusion

The aim of this chapter is to show structural models on pricing credit risky bonds. In particular, we demonstrate how the contingent claims framework provides an appropriate analytical formulation for modelling empirical regularities and analyzing the effect of credit risk on corporate bonds.

We give a concise outline of the seminal work by Merton (1974) [94] to illustrate the basis of contingent claim analysis and as a reference point for the pricing of credit risk in the structural model framework. The recent literature has shown various improvements on this original work that include first-passage time default, deviation from absolute priority, stochastic interest rates and bankruptcy costs. We show a stochastic default boundary model that capture these features. In addition, we show variants of this model, a constant- and a deterministic default boundary model. For each model we derive a closed-form formula for the price of a credit risky bond. Moreover, each model is implemented in so far as to simulate and analyze the term structure of credit spreads and the probability of default.

For the term structure of credit spreads we observe that the patterns predicted by the structural models are generally consistent with the shapes shown by empirical findings. As expected the deterministic default boundary model shows higher spreads than the constant default boundary model. However, contrary to expectation the two-factor stochastic default boundary model shows lower spreads relative to both the one-factor constant- and deterministic default boundary models. Nonetheless, emerging studies show that credit risk is not the only determinant of credit spreads. Other factors that affect spread levels include market conditions, leverage, term structure slope, spot rate, firm growth opportunities, stock return momentum and idiosyncratic volatility.

In a similar study to that of credit spreads we implement each of the three models to show the term structure of the probability of default. These models based on a contingent claim analysis of capital structure yields a single closed-form analytic formula for corporate default probability and is calibrated easily with selected values for model parameters. Following the model assumptions corporates default with diffusive dynamics. For varying degrees of leverage and asset volatility the term structure of default rates show uniform patterns, that is, for low-, medium- and high leverage and asset volatility corporates the term structure is upward sloping. Notably, increasing multiples of leverage the default

rates remain unaffected.

The research in this chapter has shown an encouraging inquiry towards designing and evaluating the performance of structural credit risk models. A further extension of this research would be to calibrate the models with market prices of bonds so as to validate the predictive power of each model.

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## 3.12 Appendix 3.1

### 3.12.1 A Constant Default Boundary Model

The price of the defaultable zero coupon bond is expressed as:

$$\bar{P}(t, T) = Vd \left[ 1 - \mathbb{Q}(\tau \leq T) \right] + Vbe^{-r(T-t)} \mathbb{Q}(\tau \leq T)$$

where

$$\mathbb{Q}(\tau \leq T) = N(x_1) + \left( \frac{\bar{v}}{V_t} \right)^{2a} N(x_2)$$

$$x_1 = \frac{\ln \frac{\bar{v}}{V_t} - \left( r - \frac{1}{2} \sigma_V^2 \right) (T - t)}{\sigma_V \sqrt{T - t}}$$

$$x_2 = \frac{\ln \frac{\bar{v}}{V_t} + \left( r - \frac{1}{2} \sigma_V^2 \right) (T - t)}{\sigma_V \sqrt{T - t}}$$

$$a = \frac{r - \frac{1}{2} \sigma_V^2}{\sigma_V^2}$$

$$b = \frac{\bar{v}}{V_t}$$

$$d = \frac{Fe^{-r(T-t)}}{V}$$

The following figures demonstrate the credit spread characteristics for varying the default risk measures of leverage and corporate asset volatility. Table 3.3 summarizes the simulation values for the credit spreads.



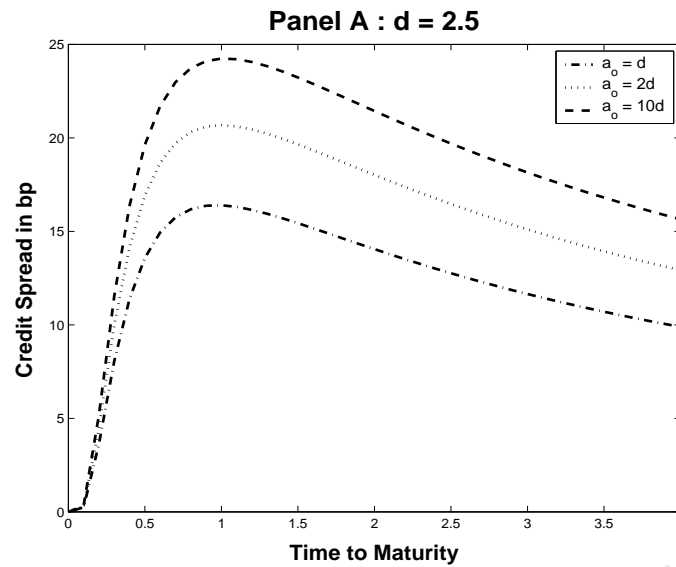


Figure 3.11: Term Structure of Credit Spreads in the Constant Default Boundary Model with varying Firm Leverage  $d = 2.5$

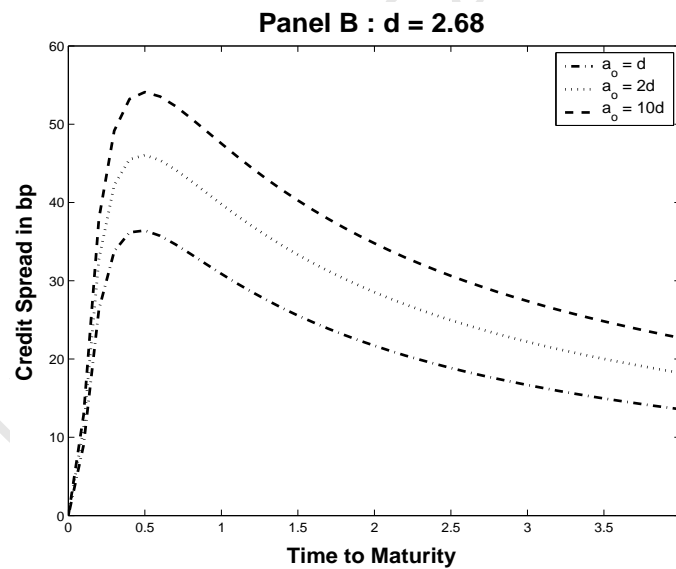


Figure 3.12: Term Structure of Credit Spreads in the Constant Default Boundary Model with varying Firm Leverage  $d = 2.68$

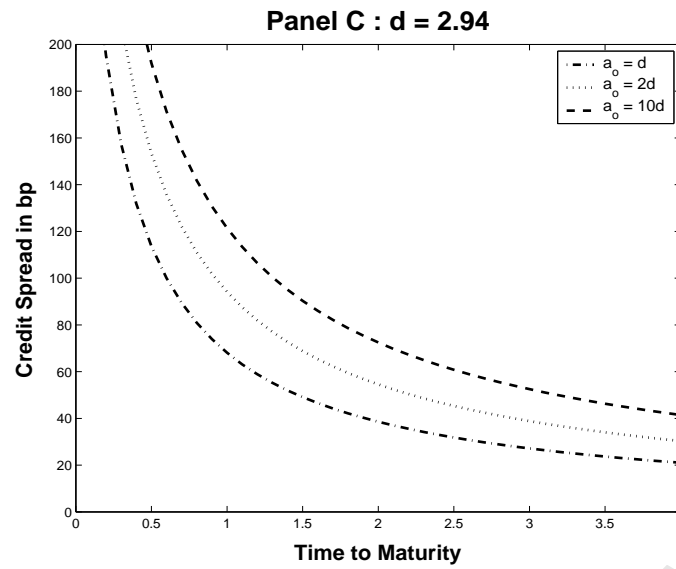


Figure 3.13: Term Structure of Credit Spreads in the Constant Default Boundary Model with varying Firm Leverage  $d = 2.94$

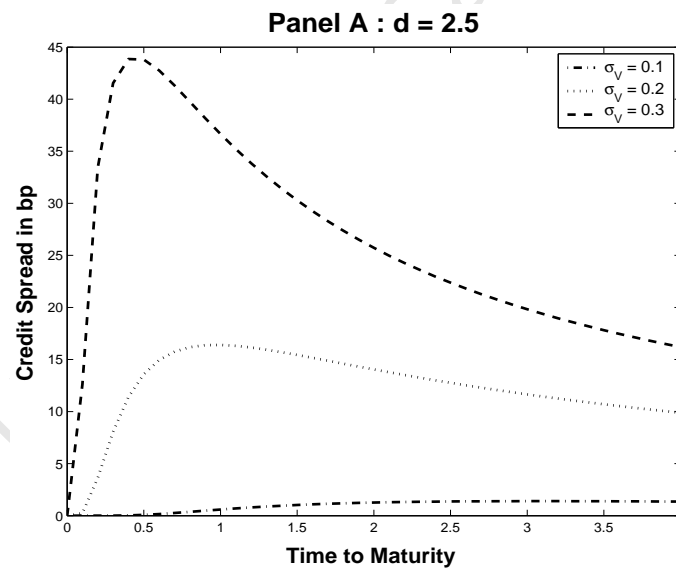


Figure 3.14: Term Structure of Credit Spreads in the Constant Default Boundary Model with varying Asset Volatility at Leverage  $d = 2.5$

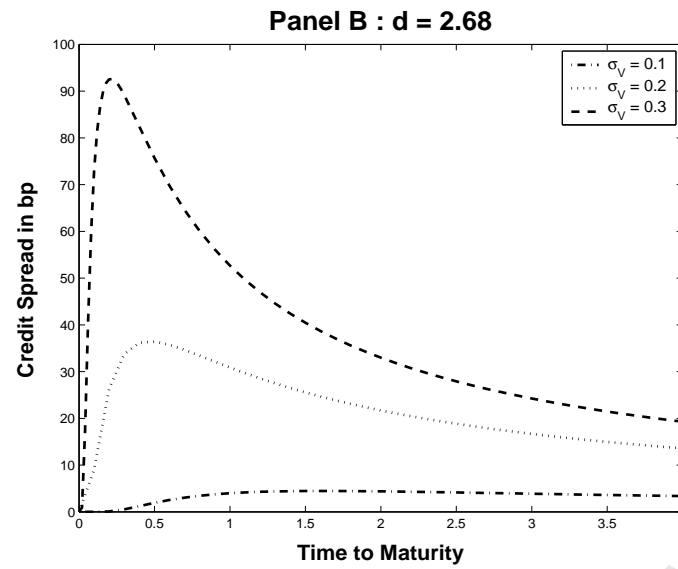


Figure 3.15: Term Structure of Credit Spreads in the Constant Default Boundary Model with varying Asset Volatility at Leverage  $d = 2.68$

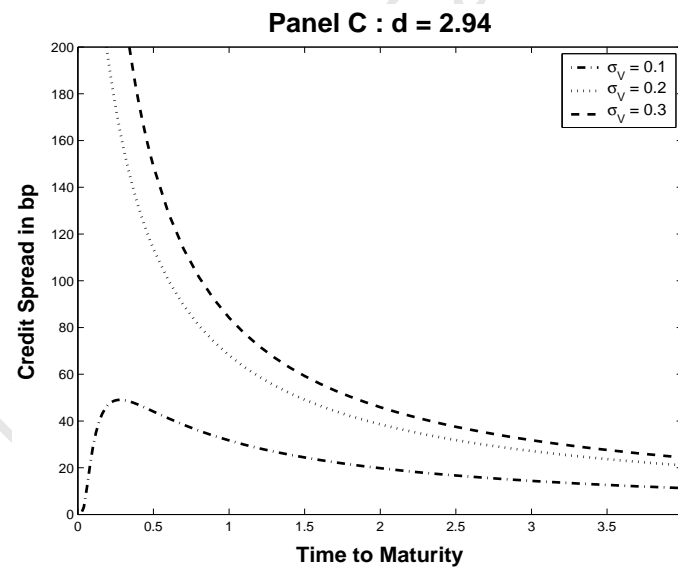


Figure 3.16: Term Structure of Credit Spreads in the Constant Default Boundary Model with varying Asset Volatility at Leverage  $d = 2.94$

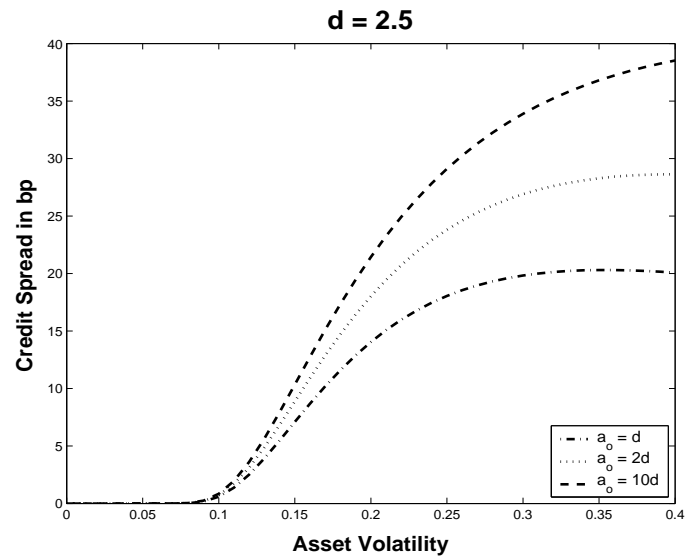


Figure 3.17: Term Structure of Credit Spreads in the Constant Default Boundary Model vs Asset Volatility at Leverage  $d = 2.5$

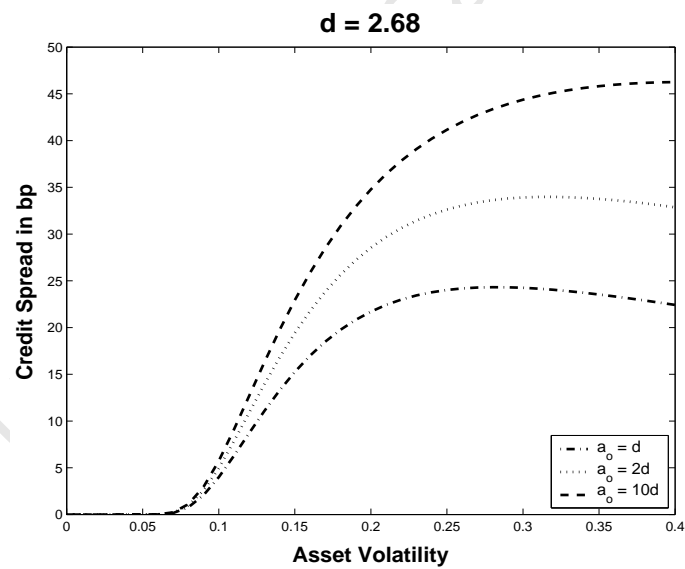


Figure 3.18: Term Structure of Credit Spreads in the Constant Default Boundary Model vs Asset Volatility at Leverage  $d = 2.68$

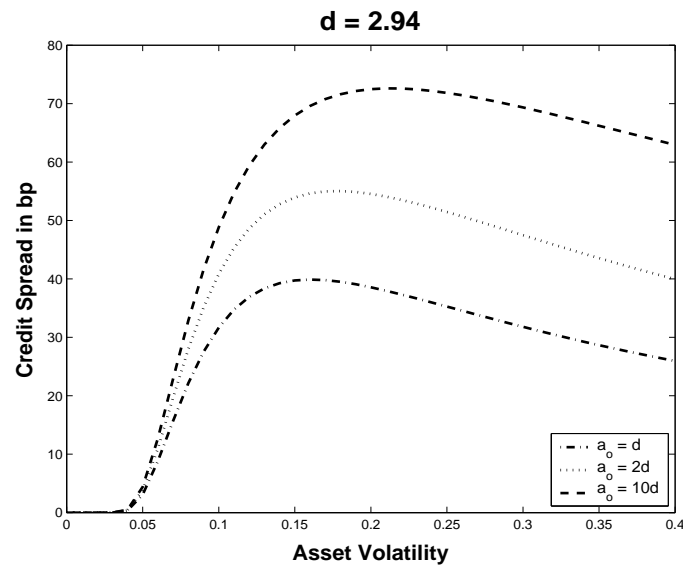


Figure 3.19: Term Structure of Credit Spreads in the Constant Default Boundary Model vs Asset Volatility at Leverage  $d = 2.94$

Time to Maturity	$d$	$a_o$ $2d$	$10d$	0.1	$\sigma_V$ 0.2	0.3
<b>Panel A : <math>d = 2.5</math></b>						
0	0.00	0.00	0.00	0.00	0.00	0.00
1	16.39	20.68	24.24	0.59	16.39	36.66
2	14.05	18.02	21.44	1.26	14.05	25.73
3	11.65	15.10	18.15	1.40	11.65	19.83
4	9.89	12.92	15.64	1.35	9.89	16.19
<b>Panel B : <math>d = 2.68</math></b>						
0	0.00	0.00	0.00	0.00	0.00	0.00
1	30.89	39.79	47.52	3.99	30.89	52.67
2	21.70	28.54	34.78	4.38	21.70	32.98
3	16.68	22.21	27.40	3.87	16.69	24.25
4	13.58	18.23	22.70	3.36	13.58	19.27
<b>Panel C : <math>d = 2.94</math></b>						
0	0.00	0.00	0.00	0.00	0.00	0.00
1	68.09	94.16	121.39	31.67	68.09	84.20
2	38.56	54.55	72.48	19.80	38.57	45.90
3	27.15	38.84	52.50	14.38	27.15	31.79
4	21.02	30.30	41.44	11.29	21.02	24.40

Table 3.3: Constant Default Boundary Credit Spread as a Function of Initial Quasi-Debt Ratio  $a_o$  and at varying Asset Volatility  $\sigma_V$ ,  $r = 0.05$ , (spreads in basis points)

### 3.12.2 A Deterministic Default Boundary Model

The price of the defaultable zero coupon bond is expressed as:

$$\bar{P}(t, T) = Vd \left[ 1 - \mathbb{Q}(\tau \leq T) \right] + Vbe^{-r(T-t)} \mathbb{Q}(\tau \leq T)$$

where

$$\mathbb{Q}(\tau \leq T) = N(x_1) + \left( \frac{\bar{v}_t}{V_t} \right)^{2\tilde{a}} N(x_2)$$

$$x_1 = \frac{\ln \frac{\bar{v}_t}{V_t} - \left( r - \gamma - \frac{1}{2} \sigma_V^2 \right) (T - t)}{\sigma_V \sqrt{T - t}}$$

$$x_2 = \frac{\ln \frac{\bar{v}_t}{V_t} + \left( r - \gamma - \frac{1}{2} \sigma_V^2 \right) (T - t)}{\sigma_V \sqrt{T - t}}$$

$$\tilde{a} = \frac{r - \gamma - \frac{1}{2} \sigma_V^2}{\sigma_V^2}$$

$$b = \frac{\bar{v}}{V_t}$$

$$d = \frac{Fe^{-r(T-t)}}{V}$$

The following figures demonstrate the credit spread characteristics for varying the default risk measures of leverage and corporate asset volatility. Table 3.4 summarizes the simulation values for the credit spreads.

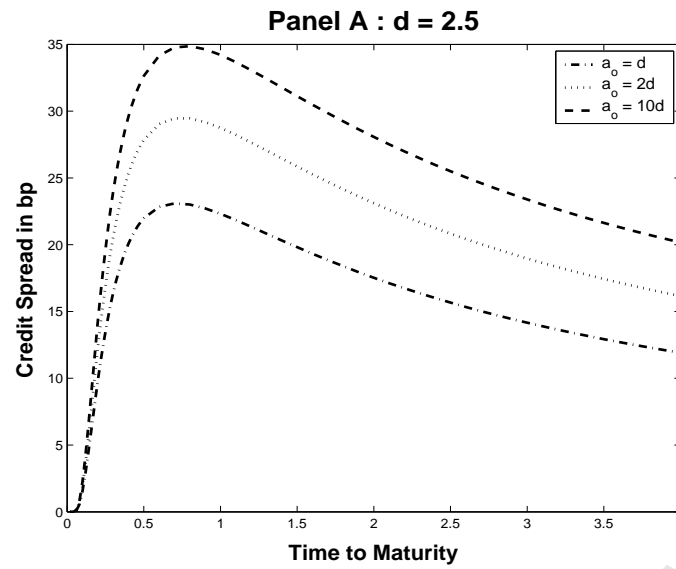


Figure 3.20: Term Structure of Credit Spreads in the Deterministic Default Boundary Model with varying Firm Leverage  $d = 2.5$

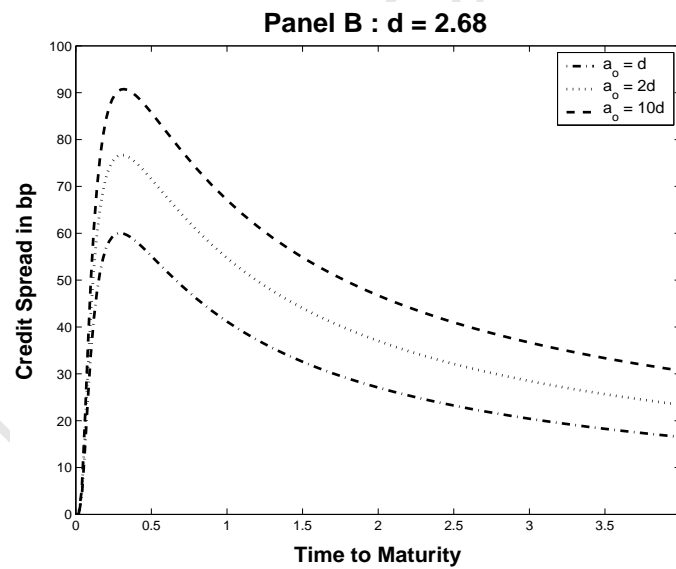


Figure 3.21: Term Structure of Credit Spreads in the Deterministic Default Boundary Model with varying Firm Leverage  $d = 2.68$

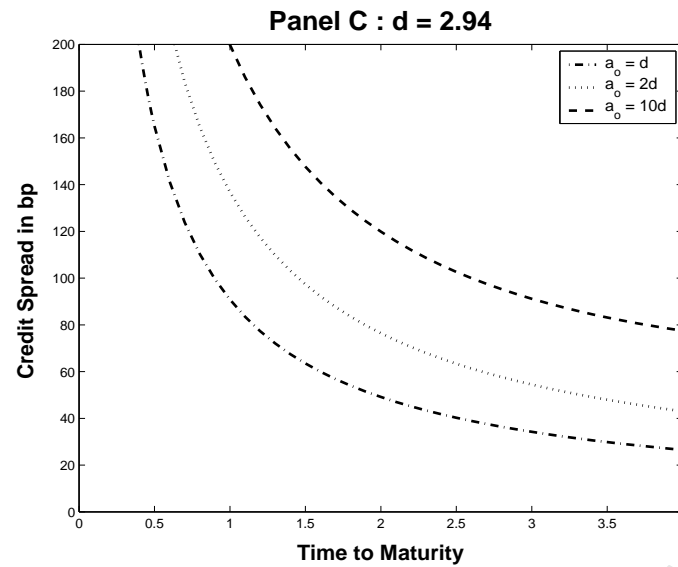


Figure 3.22: Term Structure of Credit Spreads in the Deterministic Default Boundary Model with varying Firm Leverage  $d = 2.94$

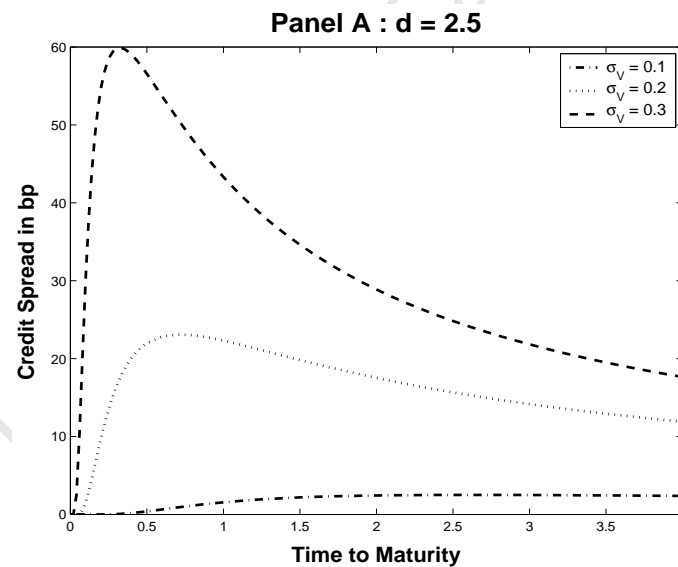


Figure 3.23: Term Structure of Credit Spreads in the Deterministic Default Boundary Model with varying Asset Volatility at Leverage  $d = 2.5$



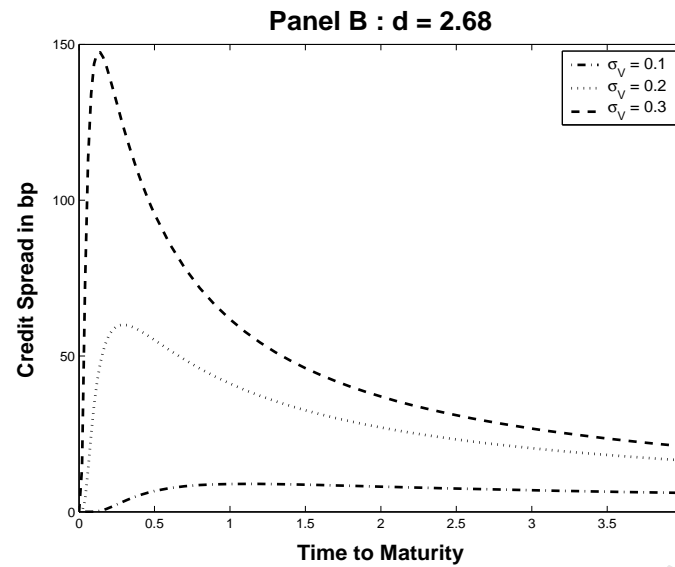


Figure 3.24: Term Structure of Credit Spreads in the Deterministic Default Boundary Model with varying Asset Volatility at Leverage  $d = 2.68$

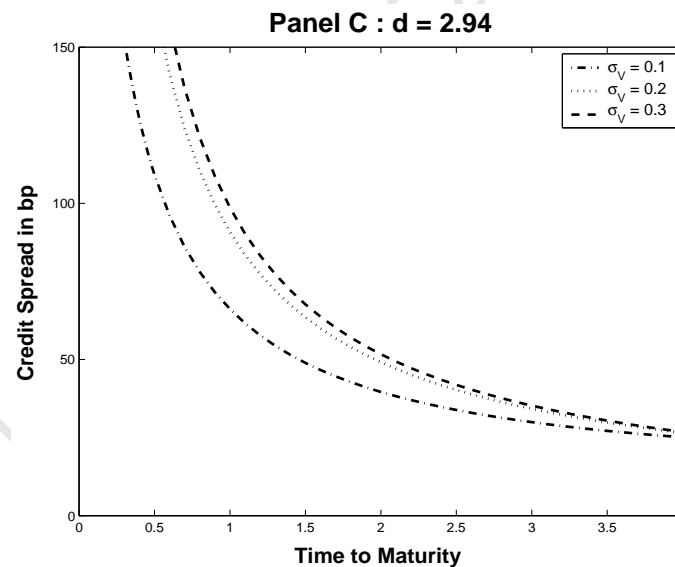


Figure 3.25: Term Structure of Credit Spreads in the Deterministic Default Boundary Model with varying Asset Volatility at Leverage  $d = 2.94$

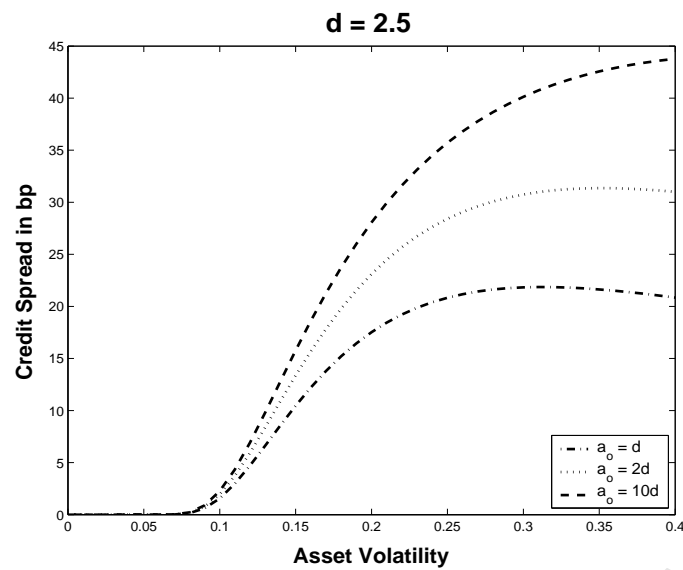


Figure 3.26: Term Structure of Credit Spreads in the Deterministic Default Boundary Model vs Asset Volatility at Leverage  $d = 2.5$

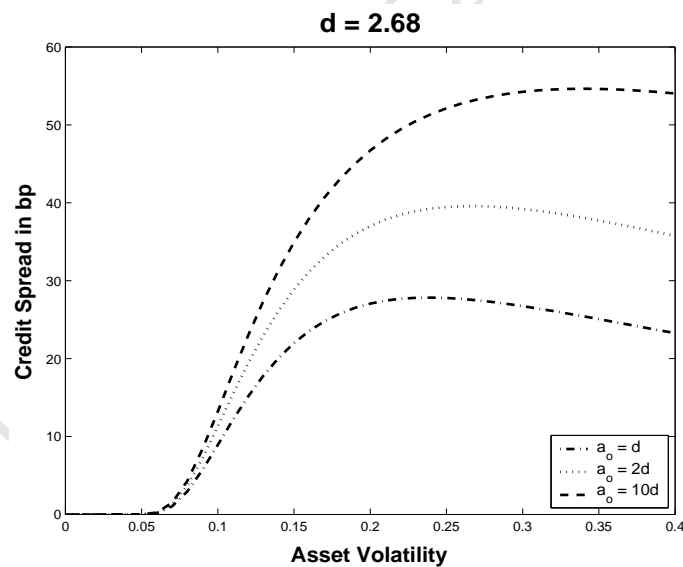


Figure 3.27: Term Structure of Credit Spreads in the Deterministic Default Boundary Model vs Asset Volatility at Leverage  $d = 2.68$

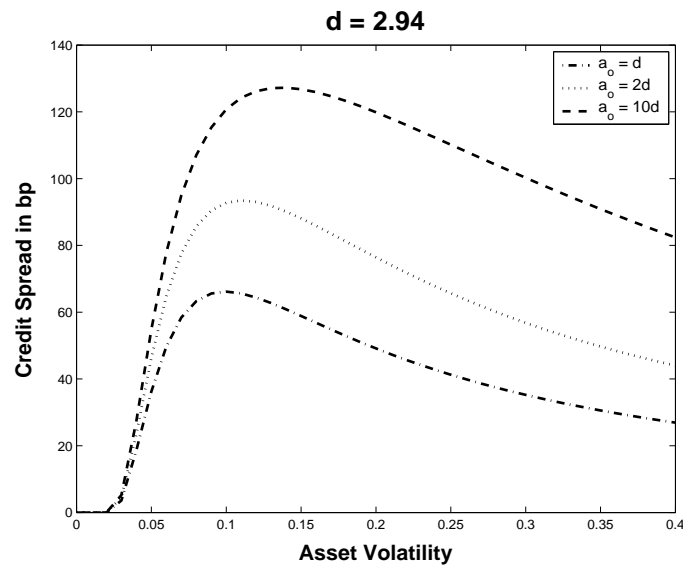


Figure 3.28: Term Structure of Credit Spreads in the Deterministic Default Boundary Model vs Asset Volatility at Leverage  $d = 2.94$

Time to Maturity	$d$	$a_o$ $2d$	$10d$	0.1	$\sigma_V$ 0.2	0.3
<b>Panel A : <math>d = 2.5</math></b>						
0	0.00	0.00	0.00	0.00	0.00	0.00
1	22.31	28.75	34.21	1.56	22.31	43.31
2	17.53	23.11	28.07	2.44	17.53	28.88
3	14.16	18.97	23.39	2.49	14.16	21.83
4	11.91	16.16	20.16	2.38	11.91	17.64
<b>Panel B : <math>d = 2.68</math></b>						
0	0.00	0.00	0.00	0.00	0.00	0.00
1	41.15	54.72	67.08	8.95	41.15	61.79
2	27.06	37.02	46.72	8.09	27.06	36.99
3	20.42	28.46	36.67	6.93	20.42	26.73
4	16.53	23.40	30.68	6.08	16.53	21.04
<b>Panel C : <math>d = 2.94</math></b>						
0	0.00	0.00	0.00	0.00	0.00	0.00
1	90.80	136.80	199.77	66.16	90.80	98.60
2	49.14	76.42	119.87	39.62	49.14	51.68
3	34.23	54.46	91.17	29.93	34.23	35.23
4	26.51	43.02	77.51	25.12	26.51	26.80

Table 3.4: Deterministic Default Boundary Credit Spread as a Function of Initial Quasi-Debt Ratio  $a_o$  and at varying Asset Volatility  $\sigma_V$ ,  $r = 0.05$ , (spreads in basis points)

### 3.12.3 A Stochastic Default Boundary Model

The price of the defaultable zero coupon bond is expressed as:

$$\bar{P}(t, T) = P(t, T) \left( 1 - \kappa (1 - \mathbb{Q}^T[\tau \geq T]) \right)$$

where  $P(t, T)$  is the risk-free zero coupon bond and  $\mathbb{Q}^T$  is just the  $T$ -forward risk neutral measure and the probability of default is calculated as :

$$\mathbb{Q}^T[\tau \geq T] = N(x_1) - e^{-2y} N(x_2)$$

where

$$x_1 = \frac{y + \frac{1}{2}\nu(T)}{\sqrt{\nu(T)}}$$

$$x_2 = \frac{-y + \frac{1}{2}\nu(T)}{\sqrt{\nu(T)}}$$

with  $y$  and  $\nu(t)$  are expressed as:

$$y = \ln \frac{V_0}{P(0, T)\gamma}$$

$$\nu(T) = \sigma_v^2 T - \rho \sigma_v \sigma_r T^2 + \frac{1}{3} \sigma_r^2 T^3$$

We now have a complete term structure of defaultable zero-coupon bonds.

The following figures demonstrate the credit spread characteristics for varying the default risk measures of leverage and corporate asset volatility. Table 3.5 summarizes the simulation values for the credit spreads.

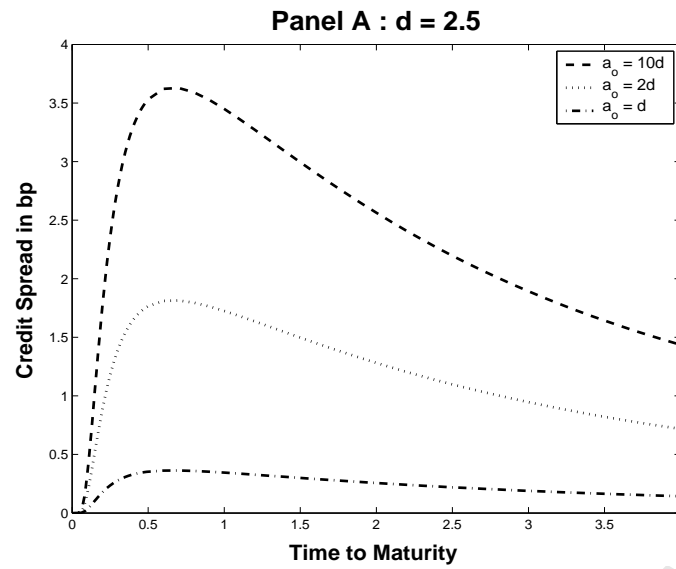


Figure 3.29: Term Structure of Credit Spreads in the Stochastic Default Boundary Model with varying Firm Leverage  $d = 2.5$

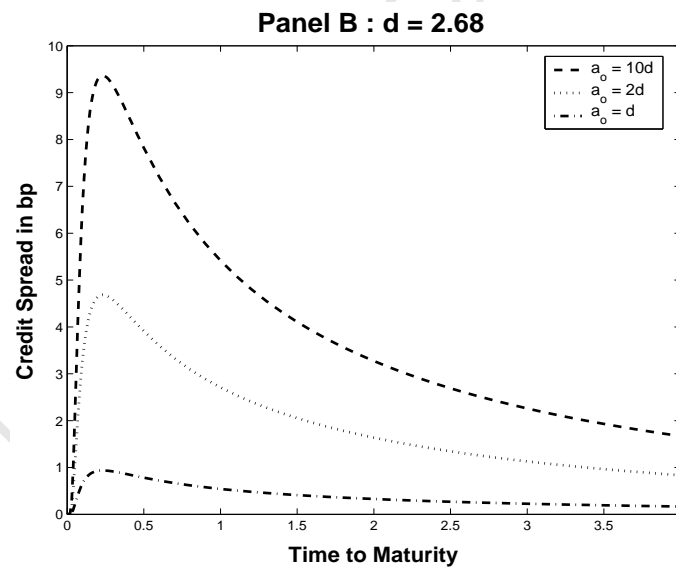


Figure 3.30: Term Structure of Credit Spreads in the Stochastic Default Boundary Model with varying Firm Leverage  $d = 2.68$

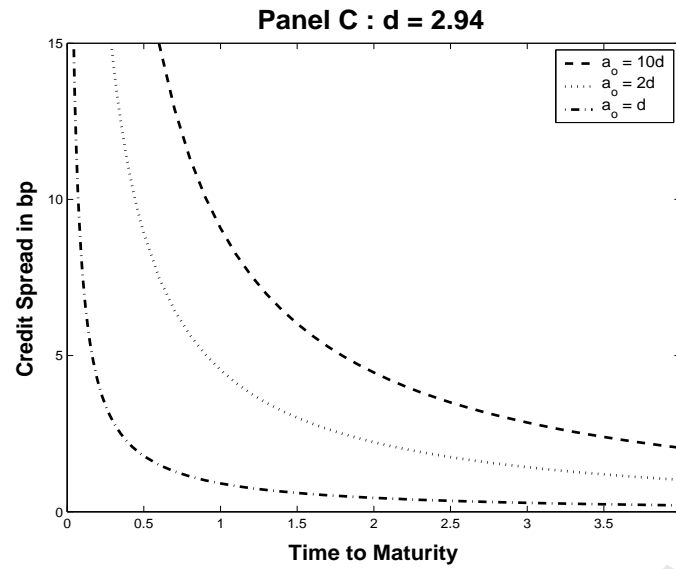


Figure 3.31: Term Structure of Credit Spreads in the Stochastic Default Boundary Model with varying Firm Leverage  $d = 2.94$

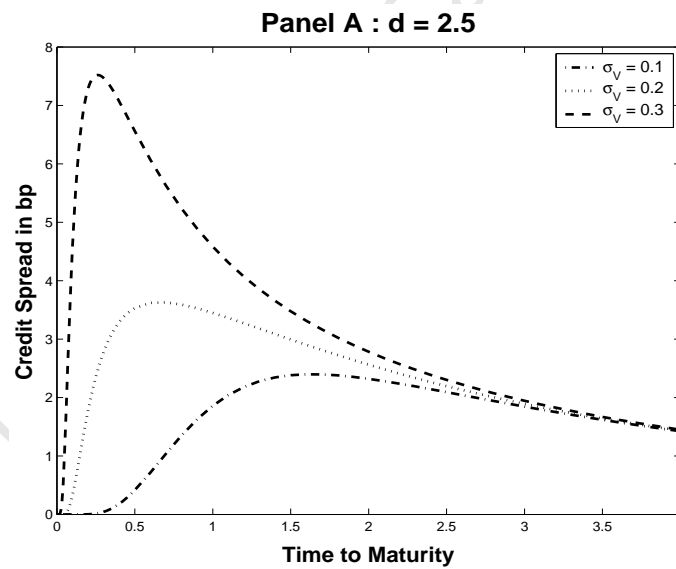


Figure 3.32: Term Structure of Credit Spreads in the Stochastic Default Boundary Model with varying Asset Volatility at Leverage  $d = 2.5$

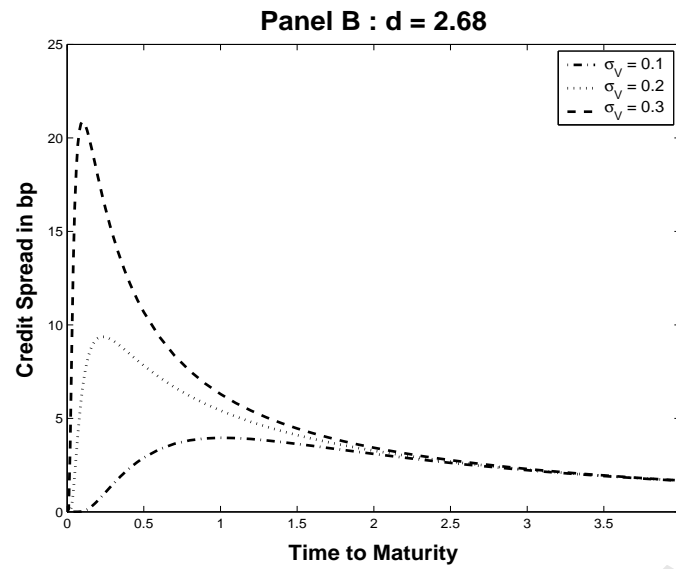


Figure 3.33: Term Structure of Credit Spreads in the Stochastic Default Boundary Model with varying Asset Volatility at Leverage  $d = 2.68$

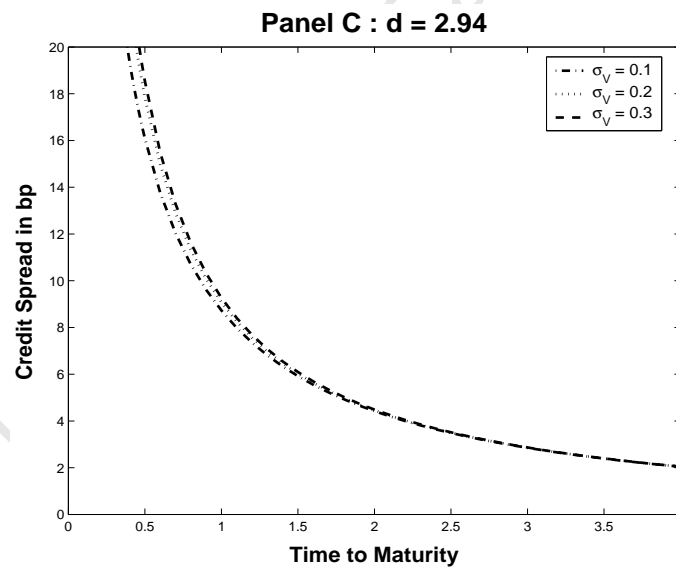


Figure 3.34: Term Structure of Credit Spreads in the Stochastic Default Boundary Model with varying Asset Volatility at Leverage  $d = 2.94$

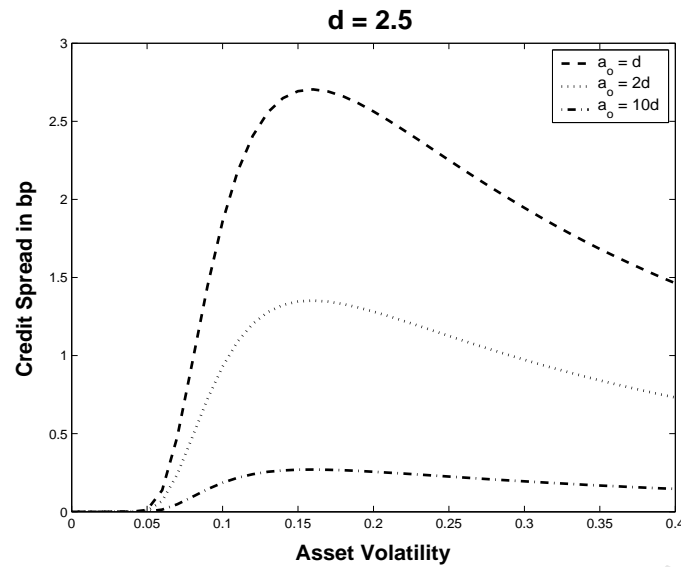


Figure 3.35: Term Structure of Credit Spreads in the Stochastic Default Boundary Model vs Asset Volatility at Leverage  $d = 2.5$

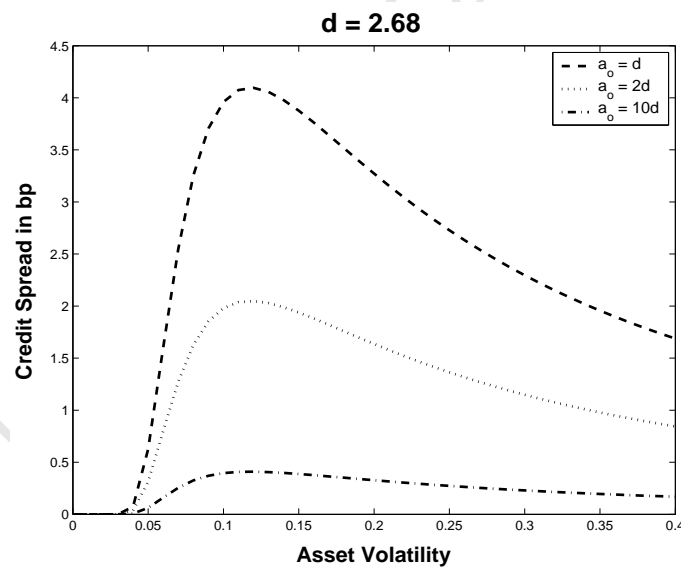


Figure 3.36: Term Structure of Credit Spreads in the Stochastic Default Boundary Model vs Asset Volatility at Leverage  $d = 2.68$



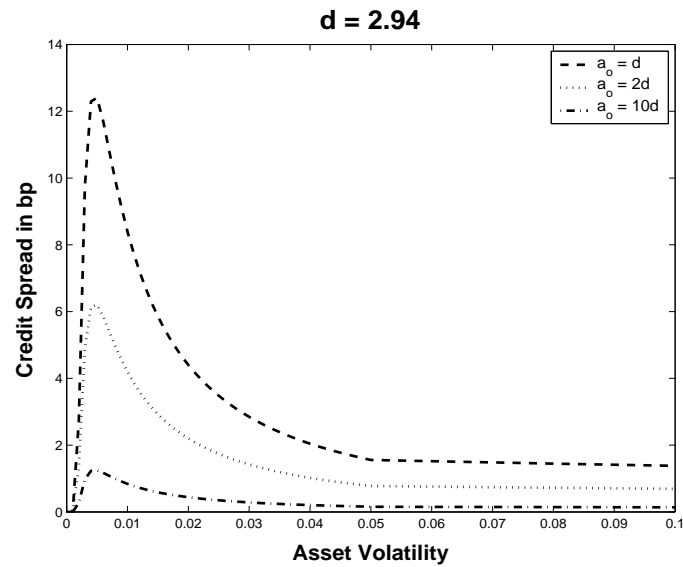


Figure 3.37: Term Structure of Credit Spreads in the Stochastic Default Boundary Model vs Asset Volatility at Leverage  $d = 2.94$

Time to Maturity	$d$	$a_o$ $2d$	$10d$	$\sigma_V$ 0.1	$\sigma_V$ 0.2	$\sigma_V$ 0.3
<b>Panel A : d = 2.50</b>						
0	0.00	0.00	0.00	0.00	0.00	0.00
1	3.45	1.72	0.34	0.10	3.45	3.94
2	2.56	1.28	0.26	0.09	2.56	2.83
3	1.89	0.95	0.19	0.09	1.89	2.01
4	1.44	0.72	0.14	0.08	1.44	1.48
<b>Panel B : d = 2.68</b>						
0	0.00	0.00	0.00	0.00	0.00	0.00
1	5.42	2.71	0.54	3.96	5.42	6.30
2	3.27	1.64	0.33	3.10	3.27	3.43
3	2.26	1.13	0.23	2.23	2.26	2.30
4	1.67	0.83	0.17	1.66	1.67	1.68
<b>Panel C : d = 2.94</b>						
0	0.00	0.00	0.00	0.00	0.00	0.00
1	9.08	4.54	0.91	8.73	9.08	9.27
2	4.46	2.23	0.45	4.42	4.46	4.49
3	2.86	1.43	0.29	2.85	2.86	2.86
4	2.05	1.02	0.20	2.04	2.05	2.05

Table 3.5: Stochastic Default Boundary Model Credit Spread as a Function of Initial Quasi-Debt Ratio  $a_o$  and at varying Asset Volatility  $\sigma_V$ ,  $r = 0.05$ , (spreads in basis points)

### 3.13 Appendix 3.2

#### 3.13.1 A Constant Default Boundary Model

In the constant default boundary model the probability of default is expressed as:

$$\mathbb{Q}(\tau \leq T) = N(x_1) + \left(\frac{\bar{v}}{\bar{V}_t}\right)^{2a} N(x_2)$$

where

$$\begin{aligned} x_1 &= \frac{\ln \frac{\bar{v}}{\bar{V}_t} - \left(r - \frac{1}{2}\sigma_V^2\right)(T-t)}{\sigma_V \sqrt{T-t}} \\ x_2 &= \frac{\ln \frac{\bar{v}}{\bar{V}_t} + \left(r - \frac{1}{2}\sigma_V^2\right)(T-t)}{\sigma_V \sqrt{T-t}} \\ a &= \frac{r - \frac{1}{2}\sigma_V^2}{\sigma_V^2} \end{aligned}$$

and  $N(\cdot)$  is the cumulative normal distribution.

In the following we show various graphs for the probability of default with varying debt and asset volatility.

*Varying Asset Volatility at Fixed Leverage*

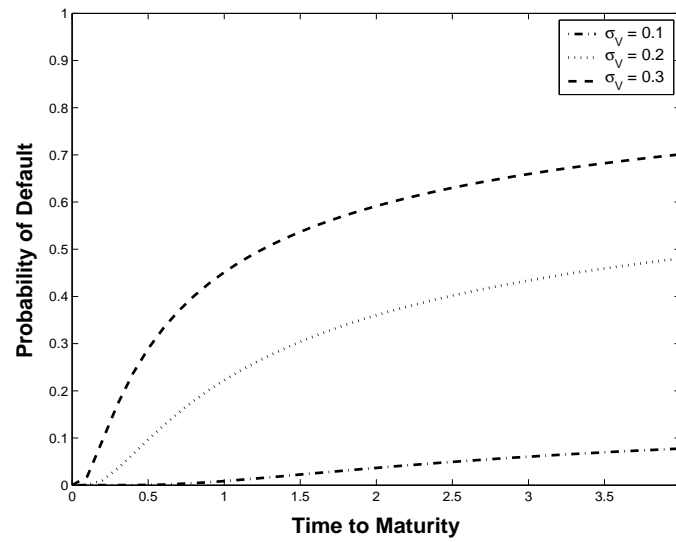


Figure 3.38: Term Structure of the Probability of Default in the Constant Default Boundary Model with varying Asset Volatility at Leverage  $d = 2.5$

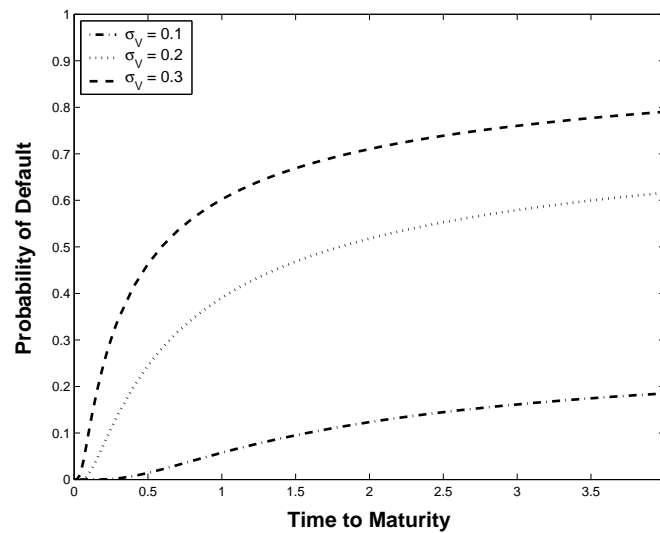


Figure 3.39: Term Structure of the Probability of Default in the Constant Default Boundary Model with varying Asset Volatility at Leverage  $d = 2.68$

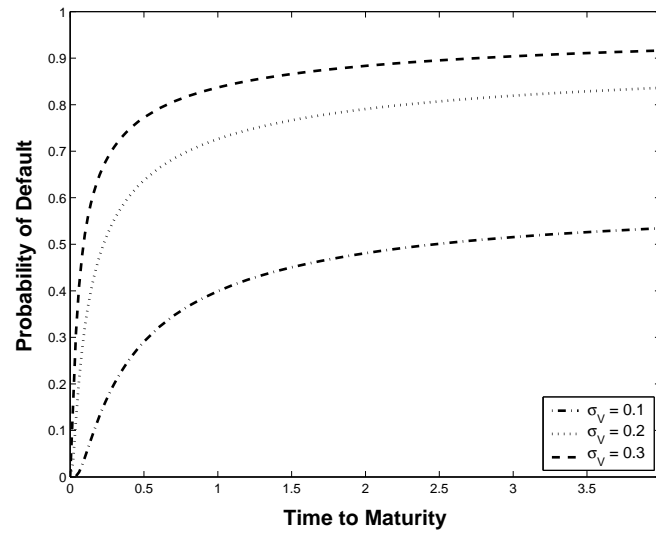


Figure 3.40: Term Structure of the Probability of Default in the Constant Default Boundary Model with varying Asset Volatility at Leverage  $d = 2.94$

*Varying Leverage vs Asset Volatility*

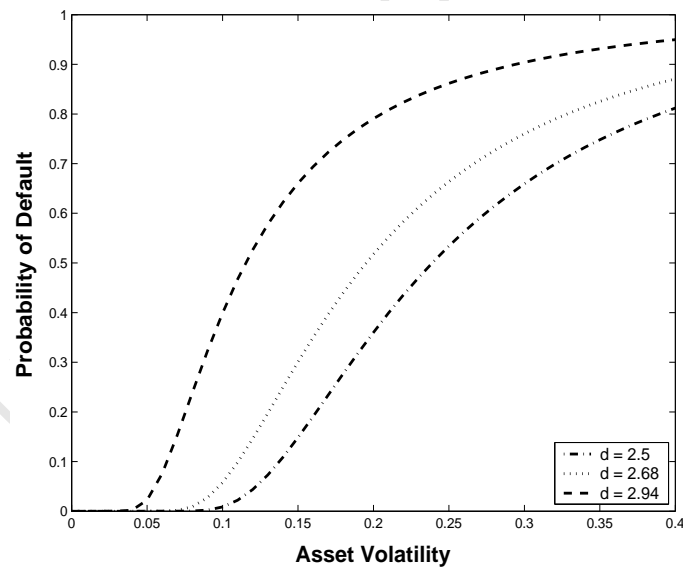


Figure 3.41: Term Structure of the Probability of Default in the Constant Default Boundary Model with varying Leverage vs Asset Volatility

### 3.13.2 A Deterministic Default Boundary Model

In the deterministic default boundary model the probability of default is expressed as:

$$\mathbb{Q}(\tau \leq T) = N(x_1) + \left(\frac{\bar{v}_t}{V_t}\right)^{2\tilde{a}} N(x_2)$$

where

$$\begin{aligned} x_1 &= \frac{\ln \frac{\bar{v}_t}{V_t} - \left(r - \gamma - \frac{1}{2}\sigma_V^2\right)(T - t)}{\sigma_V \sqrt{T - t}} \\ x_2 &= \frac{\ln \frac{\bar{v}_t}{V_t} + \left(r - \gamma - \frac{1}{2}\sigma_V^2\right)(T - t)}{\sigma_V \sqrt{T - t}} \\ \tilde{a} &= \frac{r - \gamma - \frac{1}{2}\sigma_V^2}{\sigma_V^2} \end{aligned}$$

and  $N(\cdot)$  is the cumulative normal distribution.

In the following we show various graphs for the probability of default with varying debt and asset volatility.

*Varying Asset Volatility at Fixed Leverage*

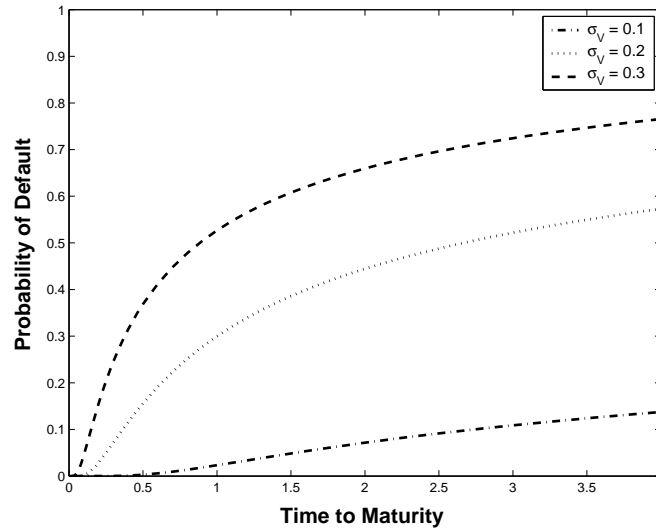


Figure 3.42: Term Structure of the Probability of Default in the Deterministic Default Boundary Model with varying Asset Volatility at Leverage  $d = 2.5$

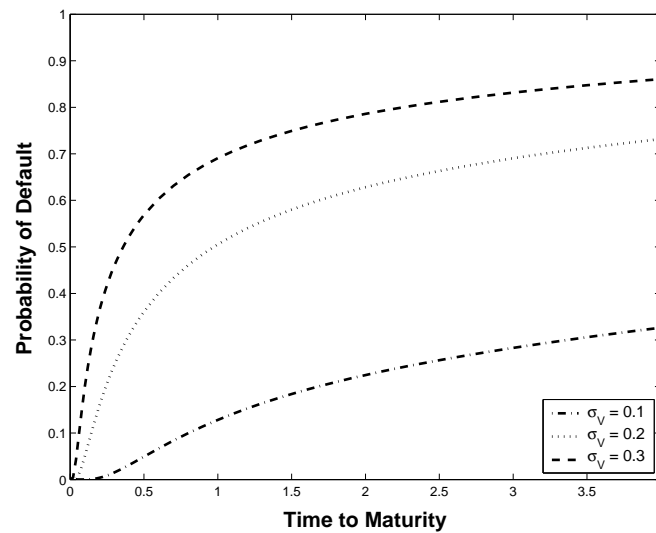


Figure 3.43: Term Structure of the Probability of Default in the Deterministic Default Boundary Model with varying Asset Volatility at Leverage  $d = 2.68$

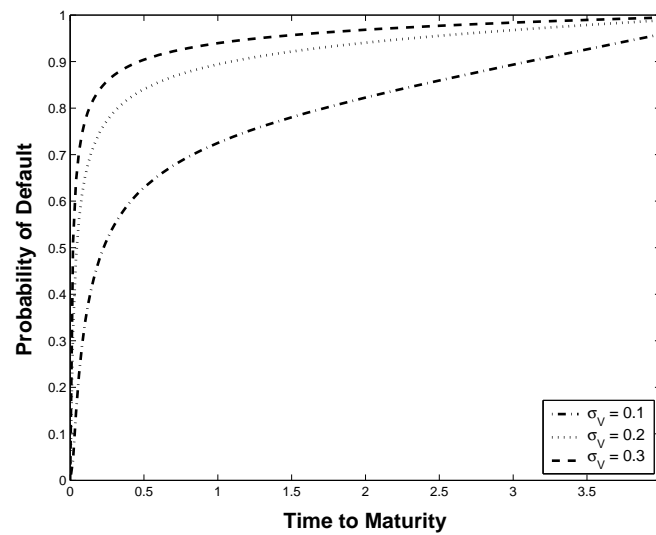


Figure 3.44: Term Structure of the Probability of Default in the Deterministic Default Boundary Model with varying Asset Volatility at Leverage  $d = 2.94$

*Varying Leverage vs Asset Volatility*

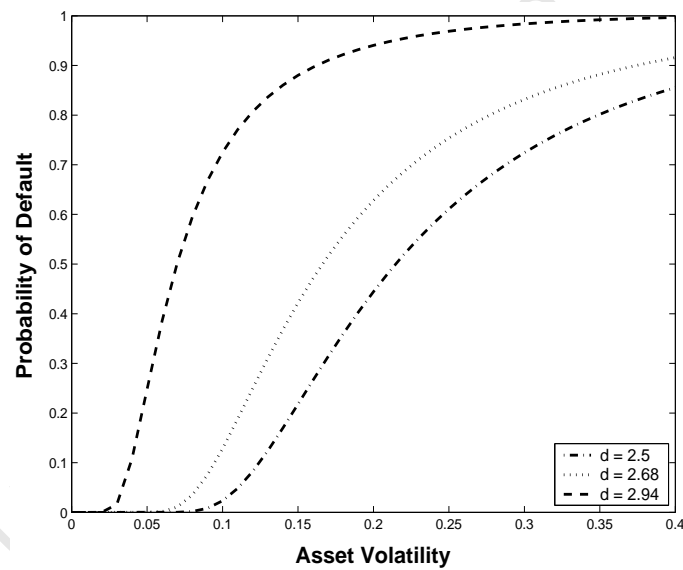


Figure 3.45: Term Structure of the Probability of Default in the Deterministic Default Boundary Model with varying Leverage vs Asset Volatility

### 3.13.3 A Stochastic Default Boundary Model

In the first passage time model with stochastic interest rates the probability of default is expressed as:

$$\mathbb{Q}^T[\tau \leq T] = 1 - N(x_1) - e^{-2y} N(x_2)$$

where

$$x_1 = \frac{y + \frac{1}{2}\nu(T)}{\sqrt{\nu(T)}}$$

$$x_2 = \frac{-y + \frac{1}{2}\nu(T)}{\sqrt{\nu(T)}}$$

with  $y$  and  $\nu(t)$  are expressed as:

$$y = \ln \frac{V_0}{P(0, T)\gamma}$$

$$\nu(T) = \sigma_v^2 T - \rho \sigma_v \sigma_r T^2 + \frac{1}{3} \sigma_r^2 T^3$$

and  $N(\cdot)$  is the cumulative normal distribution.

In the following we show various graphs for the probability of default with varying debt and asset volatility.



*Varying Asset Volatility at Fixed Leverage*

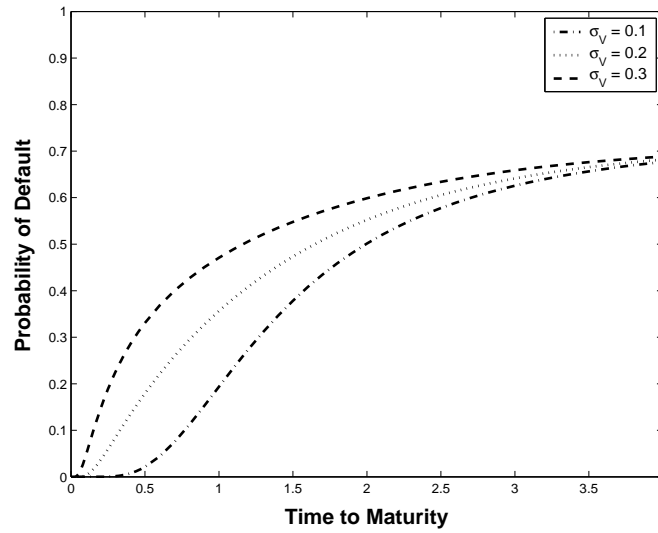


Figure 3.46: Term Structure of the Probability of Default in the Stochastic Default Boundary Model with varying Asset Volatility at Leverage  $d = 2.5$

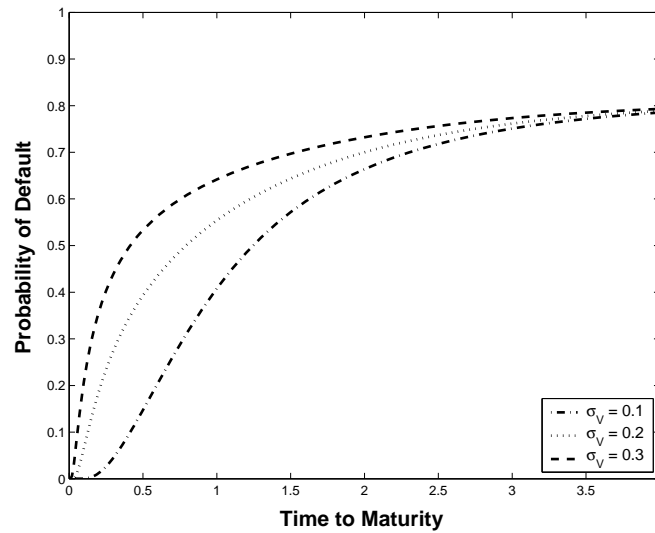


Figure 3.47: Term Structure of the Probability of Default in the Stochastic Default Boundary Model with varying Asset Volatility at Leverage  $d = 2.68$

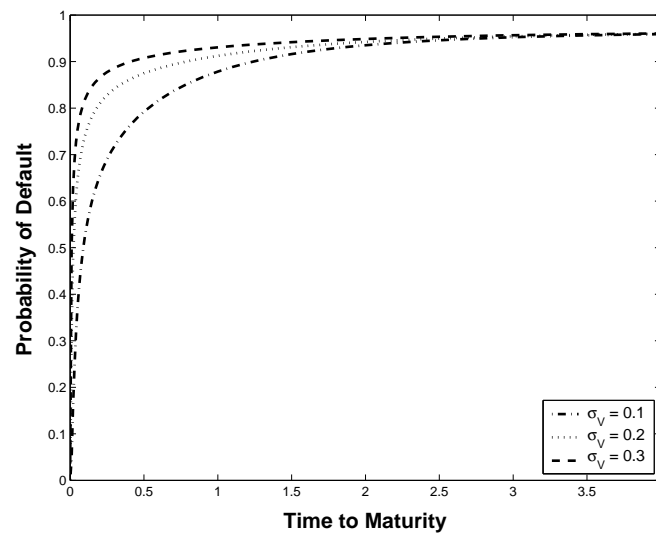


Figure 3.48: Term Structure of the Probability of Default in the Stochastic Default Boundary Model with varying Asset Volatility at Leverage  $d = 2.94$

*Varying Leverage vs Asset Volatility*

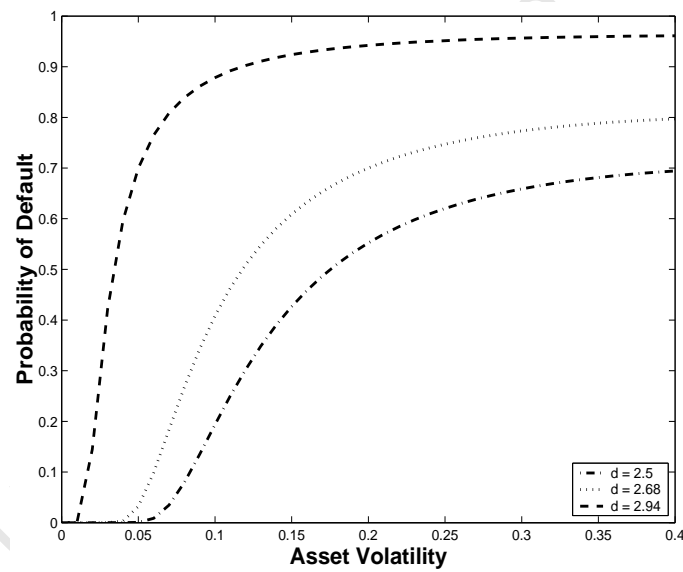


Figure 3.49: Term Structure of the Probability of Default in the Stochastic Default Boundary Model with varying Leverage vs Asset Volatility

# Chapter 4

## Reduced Form Models: Part 1

### 4.1 Introduction

In the introduction chapter we asserted that a review of the literature shows two primary categories of credit risk models in which default can be embedded in an interest rate model. Recall, the first category of credit risk models, termed structural models, was postulated by Merton (1974) [94] and based on the equity option pricing technique of Black and Scholes (1973) [20]. The basic intuition underlying the Merton (1974) [94] firm value model is that default occurs when the value of the firm's assets is lower than that of its liabilities at an arbitrarily defined maturity date.

Following the basic model by Merton (1974) [94], subsequent work by Black and Cox (1976) [18], Geske (1977) [54], and others attempt to refine the original model by Merton (1974) [94] by relaxing its strict assumptions. Notwithstanding several improvements to the original firm value model, structural models still display limitations that can be illuminated by the following observations. First, the current market value of the firm is not an easily observable process and the security price process is modelled as a proxy for the estimates of the parameters for the firm's asset value. Second, it is common practice for credit rating agencies to periodically review the credit worthiness of a corporate entity. Such changes in credit ratings are not easily incorporated in structural models. Empirical analysis of structural models by Eom, Helwege and Huang (2001) [46] suggests that credit downgrades of default risky debt regularly precedes default. Consequently, structural models display the drawback of not having an economic mechanism to compensate the model for probable credit rating changes and its associated impact on the default event. Finally, the model assumption of the evolution of the firm value in continuous time suggests that investors are explicitly able to predict the arrival of a default event. These factors promoted an opportunity for researchers to seek alternate models to price the risk of default.

In this chapter we concern ourselves with the second category of credit risk models that is widely termed reduced form models. The main thrust of reduced form models derives from the attempt to diffuse the afore-mentioned inherent limitations of structural

models. In contrast to structural models, the reduced form approach supposes that default is not contingent on the firm value and the parameters related to the structural characteristics of the value of the firm, its asset volatility and capital structure, need not be quantified for model implementation. The fundamental feature that distinguishes reduced form models from their structural model counterparts is the extent to which they can predict the default event as they are calibrated to model events to evolve with a natural unpredictability or randomness.

In credit risk modelling we are directly interested in losses associated with the default of various types of counterparty. Further, the reduced form model framework suppose the events that are modelled are characterized by stochastic dynamics. The mathematical constructs used to model such events is encompassed by point processes. A *point process* is a stochastic process whose realizations are not paths but instead counting measures. It is common to find each point assigned a specific (random) quantity and this gives way to what is known as a marked point process. This analogy is extended to credit risk modelling such that when a default occurs a loss size is assigned to the credit event. Given the above, in reduced form models, we adopt the simplest and most fundamental marked point process, the Poisson process, to model the risk of default. We say that a homogeneous Poisson process is uniquely characterized by a constant intensity or rate  $\lambda > 0$ . The intensity parameter is alternately defined as the hazard rate and as such expands the definition of an intensity process.

A traditional reduced form model specifies the time of default as an exogenously defined random variable with the probability of default modelled as the first jump of a hazard rate process. Usually the hazard rate process is defined a Poisson process and the time of default is described as a discrete jump in the level of the random variable.

**Example:** Suppose we have a portfolio of securities and the price fluctuations of each security is modelled by an independent Poisson process. More specifically the price process of each security evolves via the intensity of its associated Poisson process. In addition, suppose we place the portfolio in the arbitrage pricing theory (APT) framework such that the intensity of each security is calibrated to both systematic and firm-specific variables. To keep the illustration simple, if a particular firm declares bankruptcy then this is transformed through the intensity which results in a loss in the portfolio.

Typically, the parameters that drive the hazard rate is calibrated to some market data and the model name, reduced-form models, obtains directly from the reduction of the financial economics underlying the probability of default. In general, reduced form models take as inputs the dynamics of the default free interest rate process, the recovery rate of the default risky bonds at default as well as an intensity for the hazard rate process. To be precise, reduced form models supposes that at each time interval there exists a positive probability that a corporate can default on its outstanding debt. Further, both the probability of default and the recovery rate at default of the debt obligation can be modelled as stochastic processes. Consequently, we derive the price of credit risk from

the dynamics of these stochastic processes which also gives reduced form models a large measure of tractability and more realistic empirical performance.

There is a growing literature base that adopts the reduced form modelling framework as the basis for pricing credit risk. These models were originally postulated by Jarrow and Turnbull (1995) [73] and were subsequently studied by, for example, Jarrow, Lando and Turnbull (1997) [72], Madan and Unal (1998) [92], Duffie and Singleton (1999) [42], and Hughston and Turnbull (2001) [67]. The proliferation of research range from simple hazard rate models to more advanced modelling issues that introduce explicit assumptions on the stochastic processes that drive the probability of default and recovery rate at default. Research on reduced form models encompass models both in discrete time and continuous time. As a precursor to the discussion in this chapter we choose two examples to illustrate the diversity of modelling approaches in the reduced form model framework.

As a first example, the seminal work by Jarrow and Turnbull (1995) [73] is a discrete time model that allocate firms to particular credit risk classes, AAA, AA, etc. as dictated by their credit worthiness. The time of default is modelled as a first jump of a counting process. Then, assuming no default prior to time  $s$  the probability of default over a discrete time interval  $(s, s + \Delta t]$  is given as approximately  $\lambda(t)\Delta t$  with  $\lambda(t) > 0$  specified as an arbitrary hazard rate function. The Jarrow and Turnbull (1995) [73] model is consistent with a zero coupon treasury bond term structure and zero coupon corporate bond term structure for a specified credit class. By using risk-neutral pricing techniques they generate the distribution of the term structure of credit spreads for each credit rating class and derive the expected loss given default over  $(s, s + \Delta t]$ , which is exactly the product of the default probability and the recovery rate. Put differently, the economic interpretation of Jarrow and Turnbull (1995) [73] model is that they use observable security price data, the term structure of credit spreads, as a proxy for the markets sentiment on the credit default process that is used to replicate payoffs for credit risk contingent claims.

We note that the reduced form model construction in Jarrow and Turnbull (1995) [73] is fairly general and, in particular, allows for the hazard rate function to be specified as an arbitrary stochastic process. The second example is attributed to Lando 1998 [85] where he makes specific the notion of an arbitrary random process and defines this to be a Cox process. More commonly, a Cox process is known to be a doubly stochastic Poisson process with the hazard rate function conditioned on an array of state variables. Lando (1998) [85] states three basic building blocks for the pricing of credit risk contingent claims and associates each building block with a result. The first building block is used to illustrate the payoff of a simple contingent claim that pays some random amount  $F$  if default does not occur by the maturity date  $T$ , or else zero at default. The time  $t$  value of the contingent claim that matures at time  $T$  is expressed as :

$$\mathbb{E}_{\mathbb{Q}} \left[ e^{-\int_t^T r(u)du} F I_{\{\tau > T\}} \right] = I_{\{\tau > T\}} \mathbb{E}_{\mathbb{Q}} \left[ e^{-\int_t^T r(u) + \lambda(u)du} F \right]$$

where  $\tau$  denotes the default time.  $I_{\{\tau > t\}}$  is an indicator function representing the default event  $\{\tau > t\}$  and can take on two values 0 or 1 and is defined as  $\{\tau > t\} = 1$  if default

occurs at any time after  $t$ , and zero otherwise.  $r(t)$  is the spot risk-less interest rate. Expression (5.4) denotes the expected discounted payoff, under the risk neutral measure  $\mathbb{Q}$ , where  $\lambda$  represents the additional premium to compensate the investor for assuming the credit risk. Similarly, this building block can be extended to accommodate different payoff processes.

The second building block considers an investment that pays a cashflow  $\delta(u)$  per unit time at time  $u$ . These cashflows are paid in continuous time up until the maturity date  $T$ , provided default has not occurred, else zero at default. The time  $t$  value of the investment is expressed as:

$$\mathbb{E}_{\mathbb{Q}} \left[ \int_t^T \delta(u) I_{\{\tau > u\}} e^{-\int_t^u r(s) ds} du \right] = I_{\{\tau > t\}} \mathbb{E}_{\mathbb{Q}} \left[ \int_t^T \delta(u) e^{-\int_t^u r(s) + \lambda(s) ds} du \right]$$

Intuitively, we can describe this investment as a portfolio of contingent claims with one maturing each instant  $x$  between time  $t$  and  $T$ .

The third building block considers a contingent claim that pays an amount  $\bar{F}(\tau)$  that is commonly termed the recovery rate, if default occurs at time  $\tau$ , or zero otherwise. The time  $t$  value of this contingent claim can be expressed as:

$$\mathbb{E}_{\mathbb{Q}} \left[ \bar{F}(\tau) e^{-\int_t^{\tau} r(u) du} \right] = I_{\{\tau > t\}} \mathbb{E}_{\mathbb{Q}} \left[ \int_t^T \bar{F}(u) \lambda(u) e^{-\int_t^u r(s) + \lambda(s) ds} du \right]$$

Contingent claims with features of this nature can be used to hedge against the consequences of probable default by a counterparty.

A complementary subset of reduced form models is the formulation of the recovery rate process. To begin, we note that a special case of recovery rate modelling is zero recovery at default. We observe that in the Jarrow and Turnbull (1995) [73] model it is assumed that if default occurs on, say, a discount bond the bondholder will receive an amount equal to an exogenously defined fraction of an equivalent Treasury bond. Later, Duffie and Singleton (1999) [42] proposed a model that exogenously specifies the recovery rate at default as a fraction of the trading value of the bond just prior to default and hence the name recovery of market value obtains. Further, based on this assumption, they are able to derive closed form solutions for the value of a discount bond. Then the price for such a bond paying one unit at time  $T$  can be expressed as :

$$\bar{B}(t, T) = I_{\{\tau > t\}} \mathbb{E}_{\mathbb{Q}} \left[ e^{-\int_t^T r(u) + \lambda(u) \beta(u) du} \right]$$

where  $\beta(t)$  is defined as the expected loss at default and is given by  $\beta(t) = 1 - \bar{F}$  with  $\bar{F}$  the recovery rate process. In addition, in this model, the recovery rate process is allowed to be correlated with the hazard rate process and both processes can in turn depend on macro-economic state variables. Indeed, in literature, apart from the above concise representations of the recovery rate process there exists several more propositions on the

parameterization of the recovery rate process.

The outline of this chapter is as follows. In section 4.2 we setup the basic pricing framework for contingent claims. Recovery rate models is reviewed in section 4.3. In section 4.4 we setup the model assumptions and state the fundamental results that can be obtained for the valuation of defaultable claims based on the intensity process. We extend on the concept of a Poisson process in section 4.5 to define a Cox process to value a default risky contingent claim. Section 4.6 concludes this chapter.

## 4.2 Pricing Preliminaries

In this section we review the pricing of two types of bonds, risk-less bonds, which we denote by  $B(t, T)$ , and bonds deemed to incorporate the probability of default risk, denoted by  $\bar{B}(t, T)$ , which together with default probabilities constitute the elementary pricing constructs of credit risky instruments. For the purposes of this chapter we shall assume that the market uncertainty is modelled with a specification of a complete probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and a filtration  $(\mathcal{G}_t)_{t \geq 0}$  that represents the flow of information over time.

We assume a default free term structure of interest rates defined by a progressively measurable process  $r$  that is finite, non-negative and bounded and is commonly referred to as the short-rate process. In addition, the short-rate process has the property that for any period  $[t, T)$  an investment of one unit of currency at time  $t$ , reinvested continuously until  $u$  will yield the money market account value process given by:

$$\Gamma_t = e^{\int_t^T r_u du}$$

In keeping with risk neutral valuation principles we specify an equivalent martingale measure  $\mathbb{Q}$  defined on the probability space  $(\Omega, \mathcal{F}, \mathbb{Q})$  and the same given filtration  $(\mathcal{G}_t)_{t \geq 0}$ . This makes any security discounted by the money market account a  $\mathbb{Q}$ -martingale.

If we purchase a financial security, say a stock for  $F_0$ , and hold it for a period of time and then decide to sell it at a time  $t$ , what we purchased is now valued at an additional amount of total accumulated dividends which can depend on some specified process. Moreover, the stock price would have taken on additional values up until time  $t$ . Accordingly, we find several other securities tradeable in the market that exhibit the feature of accruing a *dividend*, for example, a bond has periodic coupon payments, and to keep terminology constant we will term these accruals *dividends*. Similarly, we can extend the concept of dividends to credit risky securities. That is, the promised dividend payable at some time  $T$  may not be passed on to the holder of the contingent claim if default occurs at some time  $s < T$ . In terms of recovery rate modelling dividends will be assigned a zero recovery rate. In this dissertation we adopt the convention that all contingent claim valuation follow an *ex-dividend* process.

Consider a default risky claim issued by a corporate with maturity  $T$  and can be defined as the pair  $(F, T)$  where  $F$  is a random variable and  $T$  is the stopping time at which  $F$  is paid. The price process  $S(t, T)$  of an arbitrary contingent claim  $F$  at time  $t$ , ( $0 \leq t \leq T$ ) is given as:

$$S(t, T) = \mathbb{E}_{\mathbb{Q}} \left( e^{-\int_t^T r(s) ds} F \middle| \mathcal{F}_t \right)$$

and  $S(t, T)$  follows a  $(\mathcal{F}_t, \mathbb{Q})$  martingale. To account for the default characteristics of a contingent claim  $F$  can be represented by two component contingent claims  $[(F, T), (\bar{F}, \tau)]$ . The pair  $(F, T)$  represents the claim  $F$  that the holder will receive at maturity  $T$  if there is no default while  $(\bar{F}, \tau)$  represents the recovery amount  $\bar{F}$  that the holder will receive at the time of default  $\tau$ . This means  $F$  can be expressed as:

$$F = FI_{\{\tau > T\}} + \bar{F}I_{\{\tau \leq T\}}$$

We can now write the price process  $S(t, T)$  as:

$$S(t, T) = \mathbb{E}_{\mathbb{Q}} \left( e^{-\int_t^T r(s) ds} (FI_{\{\tau > T\}} + \bar{F}I_{\{\tau \leq T\}}) \right)$$

and the recovery payment is assumed to be made at maturity  $T$ .

Clearly, we have thus far alluded that the dynamics of credit risky securities in the reduced form model framework is influenced by several explanatory variables. For example, if we choose to model bond dynamics then, first, we know that the term structure of the defaultable bond is driven by the associated short-rate process. Second, the yield of a corporate bond is reflected by its credit quality. In the pricing of credit risky securities the probability of default, more specifically the default intensity, is modelled as a measure of its credit quality. The third fundamental component that completes the pricing of a defaultable bond is characterized by the concept of payment at default which is widely termed recovery rate modelling. In the next section we shall study various recovery rate models in the context of a zero coupon defaultable bond.

### 4.3 Recovery Rate Models

Typically in a financial market when a contingent claim is in the default state the payoff on the defaulted securities almost always yield a non-zero value. This default value of a security is termed the recovery rate, denoted by  $\bar{F}$ , and is defined as a measure or the fractional value of a contingent claim that can be redeemed once an obligor has defaulted on his obligation. Default can occur during the tenor of a contingent claim, say, on an arbitrary time interval  $[t, T]$  with  $t \leq \tau \leq T$  where  $\tau$  is defined as the time of default.

In literature we find several models that specify the parameterizations of recovery rate schemes. We observe, in the equivalent recovery model, Jarrow and Turnbull (1995) [73] assume that at the time of default,  $\tau$ , a credit risky bond would have a recovery



rate equal to an exogenously specified fraction of equivalent non-defaultable bonds.<sup>1</sup>. Duffie and Singleton (1999) [42] expanded on the theme of recovery rates by postulating a fractional recovery of market value model. In this model they specify the recovery rate equal to an exogenous fraction of the market value of the default risky bond at the instant prior to default. Put differently, the recovery rate is specified as an equivalent amount of default risky bonds that is not in the default state. In short, a general parameterizations for recovery rate modelling is purely a theoretical construct as any defaulting entity has identifying idiosyncratic features that impacts on its yield at default. In the sections that follow we show that under different recovery rate assumptions we achieve the related pricing formulae for a default risky security.

### 4.3.1 Zero Recovery

As the first approach we consider the case of zero recovery, that is, the holder of the default risky security receives no compensation in the event of default. Zero recovery implies  $\bar{F} = 0$  and the security has a payoff  $F = FI_{\{\tau > T\}}$  at any time  $T > 0$ . The risk neutral pricing formula for a default risky security with payoff  $F$  at time  $T$  can be expressed as:

$$S(t, T) = \mathbb{E}_{\mathbb{Q}} \left( e^{-\int_t^T r(s) ds} FI_{\{\tau > T\}} \right) \quad (4.1)$$

Notice (4.1) implicitly suggests that given  $\tau$  is a stopping time we get zero recovery for all times  $t > \tau$ . The next theorem is adapted from Duffie [38] and makes precise the concept of zero recovery in an intensity based framework:

**Theorem 1:** Suppose that  $F$ ,  $r$  and  $\lambda$  are bounded and that, under  $\mathbb{Q}$ ,  $\tau$  is doubly stochastic driven by a filtration  $(\mathcal{F}_t)_{t \geq 0}$ , with intensity process  $\lambda$ ,  $\lambda > 0$ . Suppose, moreover, that  $r$  is  $(\mathcal{F}_t)$ -adapted and  $F$  is  $\mathcal{F}_T$ -measurable. Fix any  $t < T$ . Then, for  $t \geq \tau$ , we have  $S(t, T) = 0$ , and for  $t < \tau$ ,

$$S(t, T) = \mathbb{E}_{\mathbb{Q}} \left( e^{-\int_t^T r(u) + \lambda(u) du} F \right) \quad (4.2)$$

**Proof :** see Appendix 4.1.

As an example of (4.2), if we consider the case of a default-risky zero coupon bond we get:

$$\bar{B}^{ZR}(t, T) = \mathbb{E}_{\mathbb{Q}} \left( e^{-\int_t^T r(u) + \lambda(u) du} F \middle| \mathcal{G}_t \right)$$

which by the technical construct makes the pricing of default-risky zero-coupon bonds similar to the pricing of risk-less zero-coupon bonds.

Clearly, implicit in modelling zero recovery we observe that the additional premium for discounting for default is exactly the intensity. This concept of discounting at an additional factor to the short rate  $r$  was postulated by Lando (1998) [85]. The next

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<sup>1</sup>by equivalent we mean bonds with the same face value and maturity

step is to consider non-zero recovery at default time  $\tau$  if default occurs at  $t < \tau < T$ . More precisely, we will focus our attention on the widely accepted recovery mechanisms specified in literature.

### 4.3.2 Recovery of Treasury

The *recovery of treasury* (RT) model was made popular by Jarrow and Turnbull (1995) [73] and Longstaff and Schwartz (1995) [91]. It is assumed that, in the event of default, the holder of the contingent claim receives an exogenously specified fraction  $\beta$  of an otherwise equivalent risk-less bond. Then by definition, the RT mechanism is not linked to the pre-default value of the risky security. We now show the price process for a default risky contingent claim under the RT mechanism within our risk-neutral pricing framework. Together with the assumption of continuous compounding of the money market account we can express the default-risky component of the claim at maturity  $T$  as :

$$\bar{F}_T = \bar{F} e^{\int_{\tau}^T r(u) du} \quad (4.3)$$

Further, under the RT assumption we can write (4.3) as:

$$\bar{F}_T = \beta F$$

We can now write the contingent claim formula for  $F$  as:

$$F = FI_{\{\tau > T\}} + \beta FI_{\{\tau \leq T\}}$$

which results in the price process  $S(t, T)$  being expressed as:

$$\begin{aligned} S(t, T) &= \mathbb{E}_{\mathbb{Q}} \left( e^{-\int_t^T r(u) du} [FI_{\{\tau > T\}} + \bar{F}I_{\{\tau \leq T\}}] \right) \\ &= \mathbb{E}_{\mathbb{Q}} \left( e^{-\int_t^T r(u) du} [FI_{\{\tau > T\}} + \bar{F}_T I_{\{\tau \leq T\}}] \right) \end{aligned} \quad (4.4)$$

Next, let us turn our attention to the valuation of a default- risky zero-coupon bond at time  $t$  with maturity  $T$  under the RT assumption. By using theorem 1 in association with expression (4.4) and the result from the zero-recovery model we derive, as shown in Appendix 4.2, at any time  $t < T$  the pricing formula for a default risky zero-coupon bond as:

$$\bar{B}^{RT}(t, T) = \mathbb{E}_{\mathbb{Q}} \left( \beta B(t, T) + (1 - \beta) \bar{B}^{ZR}(t, T) \right) \quad (4.5)$$

where  $B(t, T)$  denotes the price of a risk-less zero-coupon bond at time  $t$  with maturity  $T$ . Notice, in this case, the price of a default-risky zero-coupon bond is less than the equivalent risk-less bond given that  $\beta$  is always non-negative.

### 4.3.3 Recovery of Face Value

In their paper Duffie and Singleton (1999) [42] show the *recovery of face value* (RFV) model. The RFV model supposes that the holder of the contingent claim is compensated an exogenously specified fraction of the promised face value. In addition, this model is adapted to the absolute priority rule. For example, all bonds carrying the same credit rating from an obligor should be paid equally in the instance of default. By implication of the absolute priority rule we assume at default the first claim against the obligor is apportioned to the cost of bankruptcy. To keep matters simple we assign the bankruptcy cost a constant amount  $c$ . Then at default the payoff on the RFV model yields an amount

$$\bar{F} = 1 - c \quad (4.6)$$

We can now write the contingent claim formula  $F$  as:

$$F = FI_{\{\tau > T\}} + (1 - c)I_{\{\tau \leq T\}}$$

which results in the price process  $S(t, T)$  being expressed as:

$$\begin{aligned} S(t, T) &= \mathbb{E}_{\mathbb{Q}} \left( e^{-\int_t^T r(u) du} [FI_{\{\tau > T\}} + \bar{F}I_{\{\tau \leq T\}}] \right) \\ &= \mathbb{E}_{\mathbb{Q}} \left( e^{-\int_t^T r(u) du} [FI_{\{\tau > T\}} + (1 - c)I_{\{\tau \leq T\}}] \right) \end{aligned} \quad (4.7)$$

Now, let us consider the valuation of a corporate zero coupon bond at time  $t$  with maturity  $T$  under the RFV assumption. An application of theorem 1 in association with expression (4.7) and together with the definition of the hazard rate function we derive, as shown in Appendix 4.3, at any time  $t < T$  the pricing formula for a corporate zero coupon bond as:

$$\bar{B}^{RFV}(t, T) = \mathbb{E}_{\mathbb{Q}} \left( e^{-\int_t^T r(u) + \lambda(u) du} F + \int_t^T (1 - c)\lambda(u) e^{-\int_t^T r(s) + \lambda(s) ds} du \right) \quad (4.8)$$

Notice, under this assumption an appropriate numerical technique is required to determine the value of the corporate zero coupon bond at default.

### 4.3.4 Recovery of Market Value

The *recovery of market value* is described by Duffie and Singleton (1999) [42] where they specify the default recovery rate such that the holder of the contingent claim receives a payoff equivalent to a fraction of the claim's pre-default market value. The recovery process is defined by:

$$\bar{F} = (1 - \beta)F_-$$

$$F_- = \lim_{t \rightarrow \tau} F(t, T)$$

where  $F_-$  is the value of the contingent claim at the instant before default and the contingent claim is priced in the time period  $[t, T]$  with  $\tau$  defined as the time of default. The expected loss in market value at any given time  $t = \tau$  is given by the constant  $\beta \in [0, 1]$ . In addition, Duffie and Singleton (1999) [42] propose substituting the risk-free short rate process  $r$  with a default-adjusted short rate process defined as:

$$\Lambda_u = r_u + \lambda_u \beta$$

and in the context of theorem 1 the price process  $S(t, T)$  can be expressed as :

$$S(t, T) = \mathbb{E}_{\mathbb{Q}} \left( e^{-\int_t^T \Lambda(u) du} F \right)$$

The above expression generates an intuitively appealing interest since the default-adjusted short rate process encapsulates both the intensity of default and the impact of expected losses at default. Notice, the value of the contingent claim is only dependent on the risk neutral intensity  $\lambda$  and the expected fractional loss rate  $\beta$  through the product  $\lambda\beta$ . The product  $\lambda\beta$  represents a *thinned* default intensity and is precisely the factor that is associated with the contingent claim's loss of market value in the state of default.

In the reduced form model framework it is commonplace to specify the intensity function and the recovery rate mechanism exogenously and not as a function of the value of the contingent claim process. This assertion applies when modelling corporate bond debt as an example of a contingent claim. However, Duffie and Singleton (1999) [42] suggest that this assertion may not hold true in general. Given that the mean loss rate  $\lambda\beta$  is exogenously specified we can apply standard term structure techniques to parameterize  $\Lambda$  instead of the risk-free rate  $r$  and as such we can directly apply the default-adjusted short rate process,  $\Lambda_u$ , to the pricing of a corporate bond.

Further, Duffie and Singleton (1999) [42] assert that the RMV model under certain technical adjustments can be extended to account for liquidity effects of the defaultable instrument being priced such that  $\Lambda_u$  can be expressed as:

$$\Lambda_u = r + \lambda\beta + \alpha$$

with  $\Lambda_u$  being defined as the *default and liquidity-adjusted* short rate process where  $\alpha$  can be specified as a stochastic process representing the associated liquidity effects on the valuation process.

If we continue with the corporate bond as an example of a contingent claim then we can express the price of this default risky security at time  $t$  with maturity  $T$  in the context of theorem 1 as:

$$\bar{B}^{RMV}(t, T) = \mathbb{E}_{\mathbb{Q}} \left( e^{-\int_t^T r(u) + \lambda(u)\beta(u) du} F \right)$$

with  $F$  denoting the pre-default market value of the corporate bond and  $\lambda\beta$  the thinned intensity of default.

### 4.3.5 Supplementary Remarks

In this section we made a reasonable attempt to give an overview of the various recovery models postulated in literature. Notwithstanding, it is fair to note a few observations from these recovery mechanisms. Now, the underlying methodology defining the different recovery mechanisms can be easily modified such that we can migrate between recovery models. The price of a non-zero coupon credit risky bond is determined by the sum of the present value of periodic coupon payments and the corresponding principal amount. We notice, both the RFV model and RT model make no provision to incorporate the coupon value into the recovery model. Nonetheless, a common practice is to assign the same recovery value to all bonds of identical seniority regardless of tenor. As noted, in Jarrow and Turnbull (1995) [73] and Duffie and Singleton (1999) [42] the parameterisation of RT model requires explicit modelling of riskless interest rate dynamics. Further, among other studies, Altman et al. (2003) [1] suggest that there is increasingly strong empirical evidence illustrating a negative correlation between the probability of default and recovery rates. As such recovery risk underscores its importance as a consequence of credit risk modelling.

While the main purpose of this dissertation is to price credit risk we have noted that recovery rate models is an important subset of credit risk management. The immediate implication of recovery risk is that a good estimate of the recovery rate will correspondingly reflect as a fair value contingent claim. It would be appropriate to conclude that the above theoretical characterization of recovery models apparently strengthens credit risk modelling methodology.

## 4.4 On the Intensity Based Valuation of Defaultable Claims

In this section we concern ourselves with the class of reduced form models that is also widely known as intensity based models. An intensity process is alternately known as a hazard rate model that is driven by an intensity parameter  $\lambda$  where  $\lambda$  is strictly non-negative. The primary utility of intensity based models is that they share similar features to interest rate modelling dynamics. The main result is that the intensity based model admits a new term to the risk-less interest rate process to discount the future cash-flows with a default adjusted interest rate. The additional term is designated the default intensity and will be a key feature when valuing a default risky zero coupon bond.

The aim of this section is to present the fundamental results that can be obtained by applying the intensity-based approach to the valuation of defaultable claims. In order to get explicit valuation formulae we assume there exists binary states of nature of default and non-default. The exposition we present is adapted from the work developed by Bielecki and Rutkowski (2000) [13] and we begin by presenting the model assumptions.

### 4.4.1 The Model Assumptions

We introduce the following assumptions that specify the random variables and processes associated with the defaultable claim:

**Assumption 1:** The default time  $\tau$  is an arbitrary non-negative random variable defined on an underlying probability space  $(\Omega, \mathcal{J}, \mathbb{P})$ , equipped with a filtration  $\mathbb{F} = (\mathcal{F}_t)_{t \in \mathbb{R}^+}$ ,  $\mathcal{F}_t \subset \mathcal{J}$ . The probability measure  $\mathbb{P}$  is interpreted as the equivalent martingale measure for the underlying securities market model. As usual  $\mathbb{P}\{\tau < +\infty\} = 1$  and we assume that for every  $t \in \mathbb{R}^+$ ,  $\mathbb{P}\{\tau = 0\} = 0$  and  $\mathbb{P}\{\tau > t\} > 0$ . For a given default time  $\tau$ , we associate a jump process  $N_t$  defined as  $N_t = I_{\{\tau \leq t\}}$  for  $t \in \mathbb{R}^+$  with  $\mathbb{N}$  the filtration generated by the process  $\mathcal{N}_t = \sigma(N_s : s \leq t)$ . The default time  $\tau$  is a  $\mathbb{J}$ -stopping time on the enlarged filtration  $\mathbb{J} = \mathbb{N} \vee \mathbb{F}$ .

**Assumption 2:** Define the compensated process

$$M_t = N_t - \int_0^{t \wedge \tau} \lambda_s ds \quad (4.9)$$

to be a  $\mathbb{J}$ -martingale such that  $\tau$  is a random variable with  $\mathbb{F}$ -intensity  $\lambda$ . Each defaultable claim is modelled by an intensity process  $N_t$  associated with a non-negative intensity  $\lambda$ .

**Assumption 3:** We define  $r$  to be the short-term interest rate process such that:

$$\Gamma_t = e^{\int_0^t r_s ds} \quad \text{for all } t \in \mathbb{R}^+$$

is the associated savings account process.

**Assumption 4:** For a maturity date  $T > 0$ , we define  $F$  to be a contingent claim, the amount of cash payable to the claimholder at time  $T$  in the event that there is no default up until the maturity date  $T$ . In addition, we assume that  $F$  is a  $\mathcal{F}_T$ -measurable random variable.

**Assumption 5:** For a maturity date  $T > 0$ , we define the process  $\bar{F}$  to be associated with the contingent claim  $F$  such that in the event of default,  $\tau < T$ ,  $\bar{F}$  models the payoff actually received by the claimholder. In addition, we assume the process  $\bar{F}$  is predictable with respect to the filtration  $\mathbb{F}$  and is commonly referred to as the recovery rate of the default risky contingent claim.

### 4.4.2 The Risk Neutral Valuation Formula

The value process  $F$  of a European default risky claim can be described by the triplet  $(F, \bar{F}, \tau)$  over an arbitrary maturity date  $T$ . First, we postulate that the default risky claim at time  $t = 0$  is given as:

$$F_0 = \Gamma_0 \mathbb{E}_{\mathbb{P}} \left( \int_{[0, T]} \Gamma_s^{-1} d\mathcal{D}_s \right) \quad (4.10)$$

where  $\Gamma_0$  is the savings account process valued at time  $t = 0$  and  $\mathcal{D}$  describes the ex-dividend process defined as:

$$\mathcal{D}_t = \int_{]0,t]} \bar{F}_s dN_s + F(1 - N_T)I_{\{t=T\}} \quad (4.11)$$

We can extend the value process given in (5.5) to a general formula for a default risky claim  $(F, \bar{F}, \tau)$  at any time  $t$  to equal:

$$F_t = \Gamma_t \mathbb{E}_{\mathbb{P}} \left( \int_{]t,T]} \Gamma_s^{-1} d\mathcal{D}_s \middle| \mathcal{J}_t \right) \quad (4.12)$$

The value process given in (4.12) is defined as the risk-neutral valuation formula. We can now state this formula equivalently to be:

$$F_t := \Gamma_t \mathbb{E}_{\mathbb{P}} \left( \int_{]t,T]} \Gamma_s^{-1} \bar{F}_s dN_s + \Gamma_T^{-1} F I_{\{T < \tau\}} \middle| \mathcal{J}_t \right) \quad (4.13)$$

In the event of no default the value process at maturity is given as  $F_T = F I_{\{T < \tau\}}$  and the risk-neutral valuation formula simplifies to:

$$F_t = \Gamma_t \mathbb{E}_{\mathbb{P}} \left( \Gamma_{\tau}^{-1} \bar{F}_{\tau} I_{\{t \leq \tau \leq T\}} + \Gamma_T^{-1} F I_{\{T < \tau\}} \middle| \mathcal{J}_t \right) \quad (4.14)$$

or expressed explicitly in terms of the savings account process as:

$$F_t = \mathbb{E}_{\mathbb{P}} \left( e^{-\int_0^{\tau \wedge T} r_s ds} (\bar{F}_{\tau} I_{\{t \leq \tau \leq T\}} + F I_{\{T < \tau\}}) \middle| \mathcal{J}_t \right) \quad (4.15)$$

The methodology for the intensity based approach does not suppose that a default risky claim can be attainable by trading in default-free securities. The standard arguments to postulate the existence of a replicating strategy is not generally valid for a default risky claim in the intensity-based framework. Consequently, we derive utility from the fact that the risk-neutral valuation of a default-risky claim can be supported by no-arbitrage arguments in an intensity-based model approach. In other words, a default risky claim can be priced as if it were a default-risk free claim provided that the credit spread associated with the default risky claim is imputed in the risk premium.

The following theorem provides another representation for the price process  $F$  of a default risky claim. This result is due to Duffie and Singleton (1999) [42].

**Theorem 2:** For a given  $\mathbb{F}$ -predictable process  $\bar{F}$  and  $\mathcal{F}_t$ -measurable random variable  $F$ , we define the process  $S$  by setting

$$S_t = \tilde{\Gamma}_t \mathbb{E}_{\mathbb{P}} \left( \int_t^T \tilde{\Gamma}_u^{-1} \bar{F}_u \lambda_u du + \tilde{\Gamma}_T^{-1} F \middle| \mathcal{J}_t \right) \quad (4.16)$$

where  $\tilde{\Gamma}$  is the *savings account* corresponding to the default adjusted short-term rate  $R_t = r_t + \lambda_t$  i.e.

$$\tilde{\Gamma}_t = e^{\int_0^t (r_u + \lambda_u) du} \quad (4.17)$$

Then

$$I_{\{t < \tau\}} S_t = \Gamma_t \mathbb{E}_{\mathbb{P}} \left( \Gamma_{\tau}^{-1} (\bar{F}_{\tau} + \Delta S_{\tau}) I_{\{t < \tau \leq T\}} + \Gamma_T^{-1} F I_{\{T < \tau\}} \middle| \mathcal{I}_t \right) \quad (4.18)$$

**Proof :** see Bielecki and Rutkowski (2000) [13]

From the above theorem we are able to state the following corollary:

**Corollary 1:** Let the processes  $F$  and  $S$  be defined by (4.13) and (4.16), respectively, then

$$F_t = I_{\{t < \tau\}} \left( S_t - \Gamma_t \mathbb{E}_{\mathbb{P}} \left( \Gamma_{\tau}^{-1} I_{\{\tau \leq T\}} \Delta S_{\tau} \middle| \mathcal{I}_t \right) \right) \quad (4.19)$$

**Proof :** see Bielecki and Rutkowski (2000) [13]

A particular case of (5.11) is when  $\Delta S_{\tau} = 0$  and we get

$$F_t = I_{\{t < \tau\}} \tilde{\Gamma}_t \mathbb{E}_{\mathbb{P}} \left( \int_t^T \tilde{\Gamma}_u^{-1} \bar{F}_u \lambda_u du + \tilde{\Gamma}_T^{-1} F \middle| \mathcal{I}_t \right) \quad (4.20)$$

which is the main result that we will show in the next section through the application of a Cox process. With hindsight of (4.20) the process  $S$  given by (4.16) is designated as the pre-default value of the default risky contingent claim  $F$ . In the event of no jumps, that is,  $\Delta S_{\tau} = 0$ , we get the continuity condition and with additional restrictions on the underlying filtrations we find ourselves modelling the classic case of a Cox process which will form the basis of our discussion of our section.

## 4.5 Cox Process and a Defaultable Claim

In this chapter we have so far modelled a credit risky bond in the reduced form framework using a homogeneous Poisson process with an intensity function  $\lambda > 0$  defined as constant. We extend the Poisson process by allowing the arrival rate at time  $t$  to be a function of  $t$  which by definition is a non-homogenous Poisson process. A Cox process is a generalization of a non-homogenous Poisson process and we let the intensity be a function of a random variable. In this section we proceed to show how we can apply a Cox process to the pricing of a credit risky bond.

### 4.5.1 Default Time and a Cox Process

In his paper Lando (1998) [85] describes a Cox process and extends this concept to price credit risky securities. Also, in their overview Jeanblanc and Rutkowski (1999) [74] introduce a Cox process in the modelling of default risk. We make use of both papers to model default time with a Cox process.



Recall that default is modelled by a counting process with a single jump in the time interval  $t \in [0, T]$ . The jump process was described by the indicator function:

$$N(t) = I_{\{\tau \leq t\}} \quad (4.21)$$

where the stopping time  $\tau$  is at the default event and  $N(t)$  is driven by an intensity,  $\lambda > 0$ , that is constant in a standard Poisson process. In this section we describe a jump process that has a varying intensity and the Poisson process is non-homogeneous.

First, we assume that we are given a  $n$ -dimensional stochastic process  $(X_t, t \geq 0)$  defined on an underlying filtered probability space  $(\Omega, \mathcal{V}, \mathbb{V}, \mathbb{P})$  with  $\mathbb{V} = (\mathcal{V})_{t \geq 0}$ . The model economy is driven by state variables that are deemed to be an indicator of the probability of default. The process  $X_t$  is assumed to model the dynamics of these state variables that may include the yield on government bonds, the current budget deficit, the gross domestic product forecast and the prevalent mood of investor confidence in the economy as measured by credit ratings.

Next, we define the stochastic intensity for a non-negative continuous function of the form

$$(\lambda_t)_{t \geq 0} = \lambda(X_t) \quad (4.22)$$

for some function  $\lambda : \mathbb{R}^n \rightarrow \mathbb{R}^n$ . Further, we assume we are given a random variable  $\xi$ , independent of  $X$  with an exponential probability distribution

$$\mathbb{P}(\xi \geq t) = e^{-t} \quad (4.23)$$

under the probability measure  $\mathbb{P}$ . By definition we state that the default time  $\tau$  is the first jump time of a Cox process with intensity process of the form  $\lambda(X_t)$ . Hence the canonical construction of the default time  $\tau$  corresponding to the first time when the process  $\int_0^t \lambda(X_u) du$  is above the random level  $\xi$  is expressed as:

$$\tau = \inf \left\{ t \geq 0 : \int_0^t \lambda(X_u) du \geq \xi \right\} \quad (4.24)$$

The random variables  $\xi$  and  $X$  are mutually independent and as such the associated filtration of each random variable does not have to be enlarged. The intuitive description of (4.24) is that when the integrated intensity function grows sharply and reaches the absorption state of the independent exponential random variable quicker we have the probability of the default time being small sharply increasing.

The definition of the default time leads us to the following important relationships for the conditional distribution function of  $\tau$  given the  $\sigma$ -algebra  $\mathcal{V}_t$  is for  $t \geq s$

$$\mathbb{P}(\tau > s | \mathcal{V}_t) = e^{-\int_0^s \lambda(X_u) du} \quad (4.25)$$

We proceed to show (4.25) as set out by Jeanblanc and Rutkowski (1999) [74]. We know the equality  $\{\tau > s\} = \{\int_0^s \lambda(X_u)du < \xi\}$ . From the independence assumption and the  $\mathbb{V}_t$ -measurability of  $\int_0^s \lambda(X_u)du$  for  $s \leq t$  we obtain

$$\mathbb{P}(\tau > s | \mathcal{V}_t) = \mathbb{P}\left(\int_0^s \lambda(X_u)du \geq \xi \middle| \mathcal{V}_t\right) = e^{-\int_0^s \lambda(X_u)du}$$

The construction of a default time  $\tau$  with these properties and its relation to the state variable process  $X_t$  allows us a second method to price a defaultable contingent claim. We illustrate these concepts in the next section when we price a default risky zero coupon bond.

### 4.5.2 Pricing A Default Risky Contingent Claim

In the previous section we noted the mutual independence of  $\xi$  and  $X_t$  which allowed us to define the default time  $\tau$  exclusively without any further operation on the filtration's economy. In this section we relax the independence assumption. We write  $U_t = I_{\{\tau \leq t\}}$  as the hazard rate process and define:

$$\mathcal{U}_t = \sigma\{U_s : 0 \leq s \leq t\}$$

as the natural filtration of the hazard rate process. We introduce the filtration  $\mathcal{W}_t = \mathcal{V}_t \vee \mathcal{U}_t$  that is a suitably enlarged filtration generated by the underlying filtration  $\mathbb{V}$  and the hazard process  $U$ . The informational setup may be summarized as follows:

$$\begin{aligned} \mathcal{V}_t &= \sigma\{X_s : 0 \leq s \leq t\} \\ \mathcal{U}_t &= \sigma\{U_s : 0 \leq s \leq t\} \\ \mathcal{W}_t &= \mathcal{V}_t \vee \mathcal{U}_t \end{aligned}$$

The model is now conditioned relative to the  $\sigma$ -algebra  $\mathcal{W}_t$  which gives us the precise information of the trajectory of the state variables and if the Cox process has experienced a jump in the period  $(s, t]$ .

We are now in a position to gain maximum utility from the specification of the default time  $\tau$  in the same setting of all previously established expressions in which the default time  $\tau$  is manifested through the default process  $U$  as expressed in terms of its intensity process  $\lambda_t = \lambda(X_t)$ .

In this setting, as developed by Lando (1998) [85], expression (4.20) can be derived in an explicit manner, without making a direct reference to the pre-default process  $S$  as postulated in Theorem 2 and Corollary 1, respectively. The following Proposition is motivated by the results in Proposition 3.1 in Lando (1998) [85]. Next, we state Proposition 1 which is taken from Bielecki and Rutkowski (2000) [13]

**Proposition 1:** Let the default time  $\tau$  be given by (4.24). Then we have

$$F_t = I_{\{t < \tau\}} \tilde{\Gamma}_t \mathbb{E}_{\mathbb{P}} \left( \int_t^T \Gamma_u^{-1} \bar{F}_u \lambda(X_u) du + \Gamma_T^{-1} F \middle| \mathcal{V}_t \right) \quad (4.26)$$

where the process  $\tilde{\Gamma}$  is the default-risk adjusted savings account given by:

$$\tilde{\Gamma}_t = e^{\int_0^t r(X_u) + \lambda(X_u) du}$$

**Proof:** see Appendix 4.4.

Proposition 1 combined with Corollary 1 shows that we have  $F_t = I_{\{t \leq \tau\}} S_t$  where the process  $S$  is given by (4.16). Recall a particular case of Corollary 1 is given as:

$$F_t = I_{\{t < \tau\}} \tilde{\Gamma}_t \mathbb{E}_{\mathbb{P}} \left( \int_t^T \tilde{\Gamma}_u^{-1} \bar{F}_u \lambda_u du + \tilde{\Gamma}_T^{-1} F \middle| \mathcal{J}_t \right)$$

which is just (4.20).

Notice, from the conjunction of Proposition 1, expression (4.26), and Corollary 1, expression (5.11), we deduce that the jump  $\Delta S_\tau$ , even though it may still be present in (5.11), is not consequential in the present setup of the application of a Cox Process to the valuation of a default risky contingent claim. In addition, as a measure of expositional differentiation we chose arbitrary underlying filtrations in section 4.4 and section 4.5, respectively. If we choose the underlying filtrations to coincide then we have exactly a second method to value a default risky contingent claim through the application of a Cox process.

## 4.6 Summary

In this chapter we presented an exposition of recent research efforts in the reduced form model framework. We concentrate our exposition on the intensity based approach. In particular, we review the results by Bielecki and Rutkowski (2000) [13] and formulate a Cox process as postulated by Lando (1998) [85] to illustrate the pricing of a default risky contingent claim.

We begin the setup of the reduced form model framework by characterizing default arrival risk. Default is described as the first jump of a point process. In addition, we define the time of default as a random variable and a basic construct for the valuation of contingent claims subject to credit risk. First, we characterize default arrival risk in terms of the distribution of the default time. This shows that in reduced form models the jump process is inextricably linked to the exogenously defined random time through the probability of default. A second method to characterize default arrival risk is the hazard rate function which is just the instantaneous probability of default.

The next step in the model framework is to make specific the notion of a jump process. We choose the class of intensity processes to model the jump process. Moreover,

the hazard rate function is alternately known as the intensity. We observe that a common intensity process is the Poisson process which has the stationary and independent increments property. We postulate that the Poisson distribution adequately satisfies the criteria to model a default event. That is, since the default time is rare and discretely countable we modelled the time of default as the time of the first jump of a Poisson process. The intensity of the jump process is calibrated to credit financial economics data such as volatility measures, exchange rates and bond yield spreads. We complete the reduced form model framework by specifying various recovery rate schemes for a contingent claim in the default state.

With the characteristics of the Poisson distribution we state the fundamental results that can be obtained for the intensity based valuation of default risky contingent claims. Next, we showed that we can extend the homogeneous Poisson process to get the property of non-stationary increments. Then as a special case of the non homogeneous Poisson process we construct a Cox process which is essentially a doubly stochastic process. Finally, we showed a second method to derive the price of a default risky claim based on a Cox process.

## 4.7 Appendix 4

### 4.1 Proof of Theorem 1

From (4.1), the law of iterated expectations, and the assumption that  $r$  is  $(\mathcal{F}_t)$ -adapted and  $F$  is  $\mathcal{F}_T$ -measurable,

$$\begin{aligned} S(t, T) &= \mathbb{E}_{\mathbb{Q}} \left( \mathbb{E}_{\mathbb{Q}} \left[ e^{-\int_t^T r(u) du} I_{\{\tau > T\}} F \middle| \mathcal{F}_T \vee \mathcal{G}_t \right] \middle| \mathcal{G}_t \right) \\ &= \mathbb{E}_{\mathbb{Q}} \left( e^{-\int_t^T r(u) du} F \mathbb{E}_{\mathbb{Q}} \left[ I_{\{\tau > T\}} \middle| \mathcal{F}_T \vee \mathcal{G}_t \right] \middle| \mathcal{G}_t \right) \\ &= \mathbb{E}_{\mathbb{Q}} \left( e^{-\int_t^T r(u) du} F e^{-\int_t^T \lambda(u) du} \middle| \mathcal{G}_t \right) \\ &= \mathbb{E}_{\mathbb{Q}} \left( e^{-\int_t^T r(u) + \lambda(u) du} F \middle| \mathcal{G}_t \right) \end{aligned}$$

### 4.2 Recovery Of Treasury

$$\begin{aligned} \bar{B}^{RT}(t, T) &= \mathbb{E}_{\mathbb{Q}} \left( e^{-\int_t^T r(u) du} [\mathbf{F} I_{\{\tau > T\}} + \bar{\mathbf{F}}_T I_{\{\tau \leq T\}}] \right) \\ &= \mathbb{E}_{\mathbb{Q}} \left( e^{-\int_t^T r(u) du} [\mathbf{F} I_{\{\tau > T\}} + \beta \mathbf{F} I_{\{\tau \leq T\}}] \right) \\ &= \mathbb{E}_{\mathbb{Q}} \left( e^{-\int_t^T r(u) + \lambda(u) du} \mathbf{F} + e^{-\int_t^T r(u) du} \beta \mathbf{F} [1 - I_{\{\tau > T\}}] \right) \\ &= \mathbb{E}_{\mathbb{Q}} \left( e^{-\int_t^T r(u) + \lambda(u) du} \mathbf{F} + e^{-\int_t^T r(u) du} \beta \mathbf{F} - e^{-\int_t^T r(u) + \lambda(u) du} \beta \mathbf{F} \right) \\ &= \mathbb{E}_{\mathbb{Q}} \left( e^{-\int_t^T r(u) du} \beta \mathbf{F} + e^{-\int_t^T r(u) + \lambda(u) du} \mathbf{F} (1 - \beta) \right) \\ &= \mathbb{E}_{\mathbb{Q}} \left( \beta B(t, T) + (1 - \beta) \bar{B}^{ZR}(t, T) \right) \end{aligned}$$

### 4.3 Recovery of Face Value

$$\begin{aligned}
\bar{B}^{RFV}(t, T) &= \mathbb{E}_{\mathbb{Q}} \left( e^{-\int_t^T r(u) du} F \right) \\
&= \mathbb{E}_{\mathbb{Q}} \left( e^{-\int_t^T r(u) du} [F I_{\{\tau > T\}} + \bar{F} I_{\{\tau \leq T\}}] \right) \\
&= \mathbb{E}_{\mathbb{Q}} \left( e^{-\int_t^T r(u) du} [F I_{\{\tau > T\}} + (1 - c)(1 - I_{\{\tau > T\}})] \right) \\
&= \mathbb{E}_{\mathbb{Q}} \left( e^{-\int_t^T r(u) + \lambda(u) du} F + e^{-\int_t^T r(u) du} (1 - c) [1 - e^{-\int_t^T \lambda(u) du}] \right) \\
&= \mathbb{E}_{\mathbb{Q}} \left( e^{-\int_t^T r(u) + \lambda(u) du} F + e^{-\int_t^T r(u) du} (1 - c) \int_t^T \lambda(u) e^{-\int_t^u \lambda(s) ds} du \right) \\
&= \mathbb{E}_{\mathbb{Q}} \left( e^{-\int_t^T r(u) + \lambda(u) du} F + \int_t^T (1 - c) \lambda(u) e^{-\int_t^T r(s) + \lambda(s) ds} du \right)
\end{aligned}$$

**4.4 Proof of Proposition 1:** This proof is taken from Bielecki and Rutkowski (2000) [13].

We notice that by virtue of (4.24) for any  $0 \leq t \leq u \leq T$  we have:

$$\mathbb{P}\{\tau > u | \mathcal{V}_T \vee \mathcal{U}_t\} = \begin{cases} e^{-\int_t^u \lambda(X_v) dv} & \text{on the set } \{\tau > t\} \\ 0 & \text{else} \end{cases}$$

where  $\mathcal{U}_t = \sigma\{U_u : 0 \leq u \leq t\}$ . Therefore

$$\begin{aligned}
F_t &= \Gamma_t \mathbb{E}_{\mathbb{P}} \left( \int_t^T \Gamma_u^{-1} \bar{F}_u \lambda(X_u) I_{\{u \leq \tau\}} du + \Gamma_T^{-1} F I_{\{T < \tau\}} \middle| \mathcal{W}_t \right) \\
&= \Gamma_t \mathbb{E}_{\mathbb{P}} \left( \int_t^T \Gamma_u^{-1} \bar{F}_u \lambda(X_u) \mathbb{P}\{\tau \geq u | \mathcal{V}_T \vee \mathcal{U}_t\} du \middle| \mathcal{W}_t \right) \\
&\quad + \Gamma_t \mathbb{E}_{\mathbb{P}} \left( \Gamma_T^{-1} F \mathbb{P}\{\tau > T | \mathcal{V}_T \vee \mathcal{U}_t\} \middle| \mathcal{W}_t \right) \\
&= I_{\{t < \tau\}} \Gamma_t \mathbb{E}_{\mathbb{P}} \left( \int_t^T \Gamma_u^{-1} \bar{F}_u \lambda(X_u) e^{-\int_t^u \lambda(X_v) dv} du \middle| \mathcal{W}_t \right) \\
&\quad + I_{\{t < \tau\}} \Gamma_t \mathbb{E}_{\mathbb{P}} \left( \Gamma_T^{-1} F e^{-\int_t^T \lambda(X_v) dv} \middle| \mathcal{W}_t \right) \\
&= I_{\{t < \tau\}} \tilde{\Gamma}_t \mathbb{E}_{\mathbb{P}} \left( \int_t^T \tilde{\Gamma}_u^{-1} \bar{F}_u \lambda(X_u) du + \tilde{\Gamma}_T^{-1} F \middle| \mathcal{W}_t \right)
\end{aligned}$$

The last equation is conditioned relative to the  $\sigma$ -algebra  $\mathcal{W}_t = \mathcal{V}_t \vee \mathcal{U}_t \subset \mathcal{V}_t \vee \sigma(\xi)$ . But the random variable  $\xi$  is independent of  $\mathcal{F}_T$  and since the  $\sigma$ -algebras  $\mathcal{V}_T$  and  $\mathcal{W}_T$  are conditionally independent given  $\mathcal{V}_t$ , the result follows.

## Chapter 5

# Reduced-Form Models: Part 2

## A Intensity-Based Numerical Example

### 5.1 Introduction

In Chapter 4 we showed Part 1 of reduced-form models where we presented a theoretical exposition of the results that can be obtained for the pricing of credit risky bonds. In this chapter we continue with the reduced-form model framework but show a specific application and the results that can be obtained through a intensity-based numerical example. A intensity-based model is a particular class of reduced-form models.

The research on credit risk modelling concerns the pricing and hedging of defaultable financial claims for which an extensive review is shown in Bielecki and Rutkowski (2002) [14], Schönbucher (2003) [113] and Lando (2004) [86] among others. Traditional reduced-form models formulate the price of credit risk from the primitives of a random default time, the recovery rate (or one minus loss given default) on a defaulted risky bond and a stochastic intensity default process. In particular, these models usually define independent, explicit assumptions for the process dynamics of both the probability of default and the recovery rate. Moreover, they generally specify an exogenous recovery rate that is uncorrelated with the probability of default.

A survey of literature shows that up until the recent past there was sparing research on the analysis of recovery rates in contrast to that of default risk. A plausible reason for this surge in recovery rate quantification is that the Basel II revised Framework Document (2004) [11] advises internal ratings based banks (IRB) to calibrate their loss given default (LGD) models to capture cyclical effects and their associated risks. More precisely, Basel II (2004) [11] requires IRB banks to use economic downturn LGDs so that capital allocations adequately reflect systematic variations corresponding to default risk over the credit horizon. The rationale underlying this requirement is that empirical regularities show that LGD are typically higher during economic downturns than usual conditions

and therefore a capital provision aimed at quantifying adequate capital reserves to offset expected losses during high default periods should encompass this observation.

Research shows that several factors influence the recovery rates of corporate bonds. For example, Acharya et al. (2007) [6] show that factors such as seniority in capital structure, quality of security of the defaulted debt and industry conditions at time of default are found to capture the density of recovery rates. Additionally, Altman et al. (2003) [1] show empirical evidence that the business cycle and macro-economic variables also impacts on recovery rates.

While a standard assumption in intensity-based credit risk models is that the recovery rate is exogenously specified as either a constant or a stochastic state variable we note that this assumption does not, however, reflect empirical observations. Emerging studies explicitly consider the link between the default intensity and the recovery rate given default. For example, Moody's (2002) [96] empirical research show a strong correlation between annual default probabilities and recovery rates, that is, recessionary years produce higher default rates with corresponding lower recovery rates. Chava et al. (2006) [25] develop a methodology for estimating the expected loss over an arbitrary time horizon by jointly modelling the probability of default and the recovery rate given default as impacted by systematic risk, among other factors. In addition, their empirical study attempts to model and explain the covariates that affect the default intensity, recovery rate given default and their correlation.

The application we formulate in this chapter is adapted from the work by Gaspar and Slinko (2005) [51] and show the primary contributions of their study to be as follows (i) the setup for a multiple default reduced-form model when the default events are modelled by a doubly stochastic Marked Poisson Process (DSMPP), where both intensity and the marked density depend on a state variable  $X$  (ii) a model for the influence of macroeconomic risks on credit spreads and (iii) simulate realistic patterns of credit spreads and default probability term structures.

This chapter is organized as follows. In section 5.2 we give a basic set of notation and definitions. The general setup and assumptions are discussed in section 5.3. In section 5.4 we introduce our market model. The simulation study is presented in 5.5. We discuss credit spreads in section 5.6. In section 5.7 we give a concise discussion of the macro-economic factors that affect credit risk. Section 5.8 concludes.

## 5.2 Notation and Definitions

It is instructive to construct the foreground context for the numerical example and such we introduce the definitions by first specifying the following filtrations.

**Notation 1.** We call the filtration generated by  $W(t)$  the *background filtration*  $(\mathcal{F}_t^W)_{t \geq 0}$ ,

and

$$\mathcal{G}^W = \bigvee_{t \geq 0} \mathcal{F}_t^W$$

is the information set containing all future and past background information. In our setup it will be assumed that all the default free processes are adapted to  $(\mathcal{F}_t^W)_{t \geq 0}$ .

**Notation 2.** The full filtration is reached by combining  $(\mathcal{F}_t^W)_{t \geq 0}$  and the filtration  $(\mathcal{F}_t^\mu)_{t \geq 0}$  which is generated by a Marked Point Process (MPP)  $\mu$

$$\mathcal{F}_t = \mathcal{F}_t^W \vee \mathcal{F}_t^\mu$$

**Notation 3.** We define the filtration generated by *all* the information concerning the background process  $X$ , and only past information on our MPP  $\mu$

$$\mathcal{G}_t^W = \mathcal{G}^W \vee \mathcal{F}_t^\mu$$

**Definition 4.** The *loss quota* is the fraction by which the promised final payoff of the defaultable claim is reduced each time of default. We denote the loss quota by  $q$ .

**Definition 5.** The *remaining value*, after all reductions in the face value of the defaultable claim due to defaults in the time interval  $[0, t]$ , is denoted as  $V(t)$ .

**Definition 6.** The *short credit spread*  $s(t)$  is defined as the difference between the defaultable and non-defaultable short rate

$$s(t) = \bar{r}(t) - r(t) \tag{5.1}$$

**Definition 7.** The *forward credit spread*  $s(t, T)$  is defined as the difference between the defaultable and non-defaultable forward rate

$$s(t, T) = \bar{f}(t, T) - f(t, T) \tag{5.2}$$

**Definition 8.**  $K$  is a *stochastic kernel* from  $\mathbb{R}_+$  to  $E$  if it is a mapping from  $\mathbb{R}_+ \times \varepsilon$  into  $\mathbb{R}_+$  such that:

- $K(\cdot, A)$  is measurable for all  $A \in \varepsilon$
- $K(t, \cdot)$  is a measure on  $E \forall t$

If  $K(t, E) = 1$ , then the kernel is called a *probability distribution*.

### 5.3 General Setup and Assumptions

In this section we develop a reduced-form model setup that extends the current literature, see, for example, Schönbucher (1998) [110], to show correlation between the intensity



parameter and recovery rate in credit risk models. We suppose that the economy is complete and there are no arbitrage opportunities. For ease of exposition we consider a formal setup that consists of a financial market defined on a fixed time interval  $[0, T]$  where the distribution of the events is described by a filtered probability space  $(\Omega, \mathcal{F}, \mathbb{Q}, (\mathcal{F}_t)_{0 \leq t \leq T})$  and  $\mathbb{Q}$  is the risk-neutral probability measure. The probability space carries a multi-dimensional Wiener process  $W$  and, in addition, a doubly stochastic Marked Poisson Process (DSMPP),  $\mu(dt, dq)$ , on a measurable marked space  $(E, \varepsilon)$  to model the default events. The filtration  $(\mathcal{F}_t)_{0 \leq t \leq T}$  is generated by  $W$  and  $\mu$ , i.e.  $\mathcal{F}_t = \mathcal{F}_t^W \vee \mathcal{F}_t^\mu$ . The following technical restrictions apply to our model.

**Assumption 1. (Default-free Bond Market and Forward Rates)** The exposition for this assumption is adapted from Bjork (1998) [16]. We assume the existence of a liquid market for a continuum of default-free zero-coupon bonds over a period of time  $t \in [0, T]$ . Let  $p(t, T)$  denote the price of a default-free zero-coupon bond at time  $t$  that pays one unit of currency at maturity  $T$ . Further, to describe the default-free bond market we use the Heath-Jarrow-Morton (HJM) (1992) [63] framework and model under the risk-neutral measure  $\mathbb{Q}$  the dynamics of the continuously compounded forward rates  $f(t, T)$  observed at time  $t$  for an instantaneous investment over the infinitesimal interval  $[T, T + dT]$ . It is important to observe that the specification of the forward rates is equivalent to the specification of all the bond prices. In particular, the price of a one unit of currency par value zero-coupon bond can be expressed as:

$$p(t, T) = e^{-\int_t^T f(t, s) ds} \quad (5.3)$$

Moreover, we can compute the value of a zero-coupon bond by discounting at forward rates instead of spot rates. In addition, if we invert (5.3) we can extract the forward rate by differentiating with respect to  $T$  to get:

$$f(t, T) = -\frac{\partial \ln p(t, T)}{\partial T}$$

Consequently, the risk-neutral dynamics of the default free forward rates are given as:

$$df(t, T) = \alpha(t, T)dt + \sigma(t, T)dW_t \quad (5.4)$$

where

$$\alpha(t, T) = \sigma(t, T) \int_t^T \sigma^*(t, s) ds \quad (5.5)$$

and  $\sigma(\cdot, T)$  is a row vector of regular enough adapted processes,  $W$  is a  $\mathbb{Q}$  Wiener process. Of special interest to us is the shortest forward rate denoted as  $f(t, t)$  and is termed the default-free short-rate such that  $r(t) = f(t, t)$ . From the no-arbitrage assumption and the fundamental relation in (5.3) we retrieve the bond price dynamics as

$$\frac{dp(t, T)}{p(t, T)} = r(t)dt + \eta(t, T)dW(t)$$

where  $\eta(t, T) = -\int_t^T \sigma(t, s)ds$ . We allow scope for price volatility of the discount bond to depend on maturity time, calendar time and the discount bond price at time  $t$ . For ease of exposition we shall consider the case of a single driving factor; the extension to a multi-factor model is conceptually straightforward.

**Assumption 2. (Defaultable Bond Market and Forward Rates)** In an analogous framework to the riskfree bond market we consider a defaultable bond market where the market consists of a continuum of corporate bonds with maturities  $T$ . Let  $\bar{p}(t, T)$  denote the price of a defaultable zero-coupon bond at time  $t$  with maturity  $T$ . The payoff at time  $T$  of the bond is given as  $V(T)$  the remaining part of the par value of the bond after all reductions due to defaults in the time interval  $[t, T]$ , i.e.  $\bar{p}(T, T) = V(T)$ . In addition, the risk-neutral price at time  $t$  of the defaultable bond with maturity  $T$  can be expressed as:

$$\bar{p}(t, T) = \mathbb{E}^{\mathbb{Q}} \left[ e^{-\int_t^T r_s ds} V(T) \middle| \mathcal{F}_t \right] \quad (5.6)$$

Further, we place the defaultable bond market in the HJM (1992) [63] framework and define the *instantaneous defaultable forward rate*,  $\bar{f}(t, T)$ , similar to its riskfree equivalent as:

$$\bar{f}(t, T) = -\frac{\partial \ln \bar{p}(t, T)}{\partial T} \quad (5.7)$$

The *defaultable short rate* is defined as  $\bar{r}(t) = \bar{f}(t, t)$ . Consequently, the price of a defaultable zero-coupon bond can be expressed in terms of the instantaneous defaultable forward rate as:

$$\bar{p}(t, T) = V(t) e^{-\int_t^T \bar{f}(t, s) ds} \quad (5.8)$$

where  $\bar{p}(t, t) = V(t)$ . The defaultable bond economy is driven by an underlying stochastic process  $X$  whose dynamics is influenced by economic variables such as interest rates, asset price indices and other macro-economic factors. The state variable  $X$  follows the risk-neutral diffusion process

$$dX_t = \alpha_X(t, X_t)dt + \sigma_X(t, X_t)dW_t \quad (5.9)$$

where  $W$  is a  $\mathbb{Q}$  Wiener process and  $\alpha_X$  and  $\sigma_X$  are real valued functions of  $t$  and  $X_t$ .

**Assumption 3. (Default-free Short Rate)** We specify the short rate process,  $r_t$ , to follow the Vasicek (1977) [118] model:

$$dr_t = \alpha(\mu - r_t)dt + \sigma dW_t$$

where  $r_t$  is the current level of the interest rate and  $W$  is a  $\mathbb{Q}$  Wiener process. The parameter  $\mu$  is the long run Gaussian interest rate. This model has the feature of mean reversion, that is, if the interest rate is lower than the long run mean,  $r_t < \mu$ , the parameter  $\alpha$  forces the drift to increase such that the short rate will trend in the direction of  $r_t$ . Conversely, if the short rate is higher than the long run mean,  $r_t > \mu$ , then the parameter

$\alpha$  forces the drift to decrease such that the interest rate will trend in the direction of  $r_t$ . The parameter  $\alpha$  is just the speed of adjustment of the short rate towards its long run value. This is an important property of the model since it forces the short rate to display characteristic market behavior.

With the Vasicek (1977) [118] short rate model we have the interest rate to be normally distributed with the unintended consequence of a positive probability of negative interest rates. Nonetheless, if we keep the parameter  $\alpha > 0$  and compute the limit of the expected rate for an arbitrary time  $T \rightarrow \infty$  we observe that the interest rate remains positive. For our application the model is generally suitable since it is tractable and amenable to Monte-Carlo simulation methods.

**Assumption 4. (Default Time)** The time of default is a stopping time  $\tau$  and is defined by

$$\tau := \inf \left\{ t : \int_t^T \nu(X_s) ds \geq \Theta \right\}$$

where  $\Theta$  is an exponential random variable of mean 1 and  $\nu$  is a non-negative process called the intensity process. The intensity is assumed to be a  $\mathcal{G}_t^W$ -adapted process and  $X_t$  is the stochastic state variable influenced by macro-economic risk factors.

**Assumption 5. (Doubly Stochastic Marked Poisson Process)** We call the Marked Point Process  $\mu$  a  $\mathcal{G}_t^W$ -doubly stochastic Marked Poisson Process (DSMPP) if there exists a  $\mathcal{G}^W$ -measurable random measure  $\nu$  on  $\mathbb{R}_+ \times E$  such that

$$P(\mu((s, t] \times B) = k | \mathcal{G}_s^W) = \frac{(\nu((s, t] \times B))^k}{k!} e^{-\nu((s, t] \times B)}, \quad a.s. \quad B \in E$$

We specify that the predictable compensator  $\nu(dt, dq)$  admits an intensity such that we can write  $\nu(dt, dq) = \nu(dq)dt$ . In addition, we construct the MPP such that its compensator is allowed to depend on our stochastic state variable  $X$  and conditional on the realization of the state variable it is  $\mathcal{G}_t^W$ -DSMPP to the extent we can write

$$\nu(dt, dq, \omega) = \nu(dt, dq, X_t), \quad \mathbb{Q} - a.s. \quad (5.10)$$

Further, we denote the compensated point process as:

$$\tilde{\mu}(dt, dq) = \mu(dt, dq) - \nu(dt, dq, X_t), \quad \mathbb{Q} - a.s$$

Theorem(1) in Appendix 5 shows that the DSMPP with compensator of the form (5.10) exists.

We can intuitively interpret the point process  $\mu$  as that of modelling extremal events that occur at discrete points in time, for example, a sovereign default on its debt obligation. In contrast to a standard counting process context these discrete events are distinguished by not being all of the same type, that is, each event has its own *mark*. In the sovereign default example, a natural mark would be the maturity of a bond issue with

the associated mark space being the non-negative real line.

**Assumption 6. (Multiple Default Setup)** In practice bankruptcy law is designed to offer various measures of relief to distressed entities. Of particular interest is the remedies applicable to debtor-creditor interactions with regard to defaulted debt. One remedy is to foreclose on the defaulted entity and distribute the proceeds among the creditors. However, Franks and Torous (1994) [48] find in their study that a Chapter 11 filing is generally preceded by informal negotiations between the affected parties. On the one hand, and as shown by Franks and Torous (1994) [48], creditors tend to gain a higher recovery rate on defaulted debt by opting to restructure the defaulted entity and allowing it to continue as a going concern. Implicit in this setup is the scenario that if the creditors continue to hold a write-down amount of old debt then they continue to assume the risk that the corporate can default on its debt. We can generalize this scenario to an arbitrary number of corporate defaults and a subsequent write-down of the original debt. With this foreground we formulate a multiple default setup.

A multiple default setup is based on the observation that whenever the obligor defaults, the corporate is not liquidated but instead re-organized. The consequence is that the par value of the claims is reduced by an amount  $q$ . As such we assume:

1. Default occurs at the following sequence of the stopping times  $\tau_1 < \tau_2 < \dots$ , where  $\tau_i$  is the time of the  $i$ -th jump of our point process.
2. At each default time  $\tau_i$  the jump size,  $q_i$ , mark or loss quota, is drawn from the mark space  $E = (0, 1)$ .
3. There is no total loss at default, i.e. the loss quota  $q_i < 1$  for all  $i = 1, 2, \dots$
4. In addition, we assume that both:
  - (i) the arrivals of default times  $(\tau_i)_{i \geq 1}$
  - (ii) the distribution of the loss quotas given default  $(q_i)_{i \geq 1}$
 depend upon our stochastic state process  $X$ .

Given that at each default time  $\tau_i$  the final claim amount is reduced by a loss quota  $q_i$  to  $(1 - q_i)$  multiplied by its previous default value, we obtain

$$V(t) = \prod_{\tau_i \leq t} (1 - q_i) \quad (5.11)$$

where  $q_i$  is the stochastic marker to the default time  $\tau_i$ .

**Assumption 7. (Market Index)** We assume a *market index* as a proxy for the systematic risk of an economy. Gaspar and Slinko (2005) [51] argue that the market index' volatility tend to increase when the market as a whole is depressed, that is, at low values of the index, and, on the other hand, the volatility decreases when the market index is high. To make this observation explicit we treat the market index as a functional of the

index level. We suppose that the market index  $I$  is the price of a traded asset and under risk-neutral dynamics satisfies the following SDE

$$dI_t = r(t)I_t dt + \gamma(t, I_t)I_t dW_t$$

where  $r$  is the short rate,  $\gamma$  is a row vector and  $W$  is a  $\mathbb{Q}$ -Wiener process. Notably, in our setup we choose the short-rate to be the Vasicek (1977) [118] interest rate model. In addition, for each entry  $\gamma_i$  the following holds

$$\frac{\partial \gamma_i}{\partial I}(t, I) < 0 \quad (5.12)$$

The inequality given in (5.12) represents the empirical regularity of periods of bear markets are associated with periods of higher volatility while bull markets are associated with lower volatilities.

**Assumption 8. (Sensitivity Measure)** We introduce a measure of sensitivity to systematic risk,  $\epsilon$ , such that  $\epsilon \in [0, 1]$ . We assume that corporates that are sensitive to systematic risks will correspondingly bear a greater loss as a result of an increase in their default intensities in contrast to entities that are weakly sensitive. Alternately, we can also consider  $\epsilon$  as a measure of the corporate's credit capacity implying that an entity with low credit capacity will tend to be more sensitive to business cycle effects than entities with larger credit capacities.

**Assumption 9. (The Default Intensity)** The mathematical building blocks of intensity models allow for the default intensity to be parameterized from macro-economic state variable processes. In contrast, the true dynamics that characterize intensity models is implicit in the exogenous specification of the default intensity. Notwithstanding, the main drawback in intensity models is in constructing good models for the default intensity to capture the macro-economic effects and as such makes the exogenous specification an appealing alternative. In addition, this set of alternatives can include specifications that extract quantitative information, such as bond yields and interest rate volatilities, from the process we aim to model as state variables. This foreground implies that good predictors of default should include the econometric specification of fundamental variables. As such we specify the default intensity to be a deterministic function of time, interest rates, market index dynamics and the sensitivity measure, viz.  $(t, r, I, \epsilon)$ . Further, we have

$$\lambda(t, r, I, 0) = \bar{\lambda} \quad \bar{\lambda} \in \mathbb{R}_+ \quad (5.13)$$

$$\frac{\partial \lambda(t, r, I, \epsilon)}{\partial \epsilon} > 0 \quad (5.14)$$

$$\frac{\partial \lambda(t, r, I, \epsilon)}{\partial I} < 0 \quad (5.15)$$

$$\frac{\partial \lambda(t, r, I, \epsilon)}{\partial r} > 0 \quad (5.16)$$

This assumption is based on the economic intuition that if a corporate is in a robust financial state then these entities may be least sensitive to business cycle effects. The properties shown in (5.13) and (5.14) is consistent with the hypothesis for the measure of sensitivity to systematic risks. More precisely, if the corporate is risk-neutral to business cycle effects then its default intensity remains unchanged. On the other hand, if a corporate is sensitive to business cycle effects then its default intensity will correspondingly vary.

The economic interpretation of the properties shown in (5.15) and (5.16) is that if the default intensity is parameterized from macro-economic variables then the probability of default is lower in economic upturns or periods of lower risk-free interest rates and higher in economic downturns or periods of higher risk-free interest rates. That is, the market index dynamics reflects the increased market randomness in recessions while the interest rate dynamics is associated with the regularity that the internal rate of return on a project will decrease when the costs of financing debt is higher in recessions.

**Assumption 10. (Loss Quota)** The conditional distribution of the loss quota is a deterministic function of time, interest rates and the market index dynamics, viz.  $(t, r, I)$ .  $K$  is a stochastic kernel from  $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow [0, 1]$  for any realization of  $(t, r, I)$ . We denote the cumulative distribution function of loss quota conditional on default as  $\tilde{K}$  such that

$$\tilde{K}(t, r, I, x) = \int_0^x K(t, r, I, dq), \quad \int_0^1 K(t, r, I, dq) = 1, \quad \forall \quad t, r, I$$

with the following property

$$\tilde{K}(t, r, I_1, x) \geq \tilde{K}(t, r, I_2, x), \quad \text{if } I_1 \geq I_2, \forall x \in \mathbb{R}$$

That is

$$\frac{\partial \tilde{K}(t, r, I, x)}{\partial I} > 0 \tag{5.17}$$

For fixed  $(t, r)$ ,  $\tilde{K}(t, r, I, x)$  stochastically dominates all the conditional distributions with parameter  $\underline{I}$ , such that  $\underline{I} \leq I$ .

In the context of a multi-default setup we can intuitively interpret the inequality in (5.17) as at the juncture of an  $i^{th}$  default where debt holders re-negotiate their loss quota so that the corporate can continue to operate as a going concern. Further, we assume that if the corporate's assets has decreased in value then a corresponding portion of assets available for distribution to debt holders has also decreased such that they inevitably accept a higher write-down on their debt holding. Moreover, their loss quota would be much less in the event of bankruptcy emphasizing their acceptance of a lower write-down.

## 5.4 A Market Model

The early academic credit risk models make the simplifying assumption that the recovery rate is an exogenously specified variable. More precisely, the loss quota quantity is a fixed proportion of the bond value. In particular, this was a crucial assumption for reduced-form models that allowed researchers to separate the probability of default from the loss quota in observed credit spreads. This assumption is, however, not representative of observed recovery rates. In general, recovery rates are stochastic and appear to have a cyclical vector. More recent credit risk pricing models have relaxed this assumption to assess the business cycle effects on the loss quota. Further, the default intensity is also affected by the business cycle and macro-economic effects.

As our knowledge of the determinants underlying the dynamics of credit risk improves so to our pricing models have evolved to capture these effects. An emerging stream of credit risk literature concerns the modelling of the recovery rate and its associated correlation with other state variable processes, for example, the intensity of default, and attempts to explain the phenomenon of low recovery rates in periods of high frequency of obligor defaults.

A survey of literature shows that several macro-economic variables have been used to calibrate the default intensity and assess the determinants of the recovery rate. Altman and Kishore (1996) [2] find that recovery rates are correlated with cyclical effects. With credit ratings being measure of the probability of default Altman (1989) [3] show evidence of a significant association between recovery rates and credit ratings in the period preceding default. The study by Frye (2000) [50] models both the probability of default and the recovery rate to depend on a systematic risk factor, defined as the state of the economy. Further, Crouhy, Galai and Mark (2000) [31] find statistical significance for loss quota deviation around a mean value that is consistent with economic cycles.

In this section we introduce the market model underlying our analysis. Our exposition is closely structured around the work by Gaspar and Slinko (2005) [51] and we often refer to this research for additional information. We aim to show the importance of considering the dependence between recovery and intensity of default and of showing the intuition underlying our results.

To have a completely specified model to simulate we need to define a function  $\gamma(I)$  for the market index volatility, a function  $\lambda(I, \epsilon)$  for the intensity and, a distribution function  $K(dq, I)$  for the loss quota.

### 5.4.1 Market Index Volatility

We have modelled the market index dynamics (see Assumption 7) as a geometric Brownian motion with its associated diffusion coefficient remaining unspecified. In this section we attempt to make precise this diffusion coefficient and define it as the market index volatility,  $\gamma$ . Now, as a general rule volatility drops when the stock markets turn

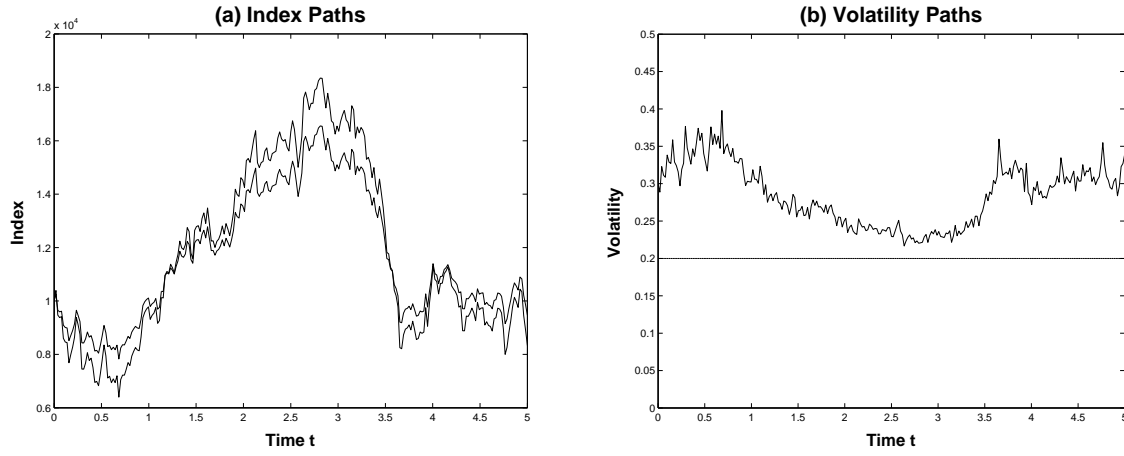


Figure 5.1: (a) Two Paths for the Index Level (b) Two Paths for the Volatility Level

bullish, from the perspective that security prices historically tended to increase with a lower momentum in contrast to when prices have decreased. In conjunction with this observation and Assumption 7, we specify the index volatility to be inversely proportional to the index level. Additionally, it is intuitively rational to state the volatility as a functional of a relative value of the market index instead of its local index value. To this extent we define a *moneyiness* ratio,  $m(I)$ ,

$$m(I) = \frac{\bar{I}}{I}$$

that relates the current value of the index to its long-run trend value where  $\bar{I}$  is exogenously specified and is regarded as the long-run trend value of the market index. Further,  $\bar{I}$  is a deterministic quantity of the risk-free rate. According to Gaspar and Slinko (2005) [51] typical levels for  $m(I)$  range from 0.7 and 1.3 where  $m = 0.7$  represents a bull market,  $m = 1.0$  represents a normal market and  $m = 1.3$  represents a bear market.

We now define the market index volatility,  $\gamma$ , as a function of the moneyiness level  $m$  as:

$$\gamma(I) = \bar{\gamma}(m(I))^{\frac{1}{2}} \quad \forall I, \bar{\gamma} \in \mathbb{R}_+ \quad (5.18)$$

where  $\bar{\gamma}$  is an arbitrarily chosen constant market index volatility.

An application of expression (5.18) is given in Figure 5.1(a) and Figure 5.1(b) where we show two possible paths for the index and volatility processes; we assume (i) the index volatility to be a constant and (ii) the index volatility is a function of the index level.



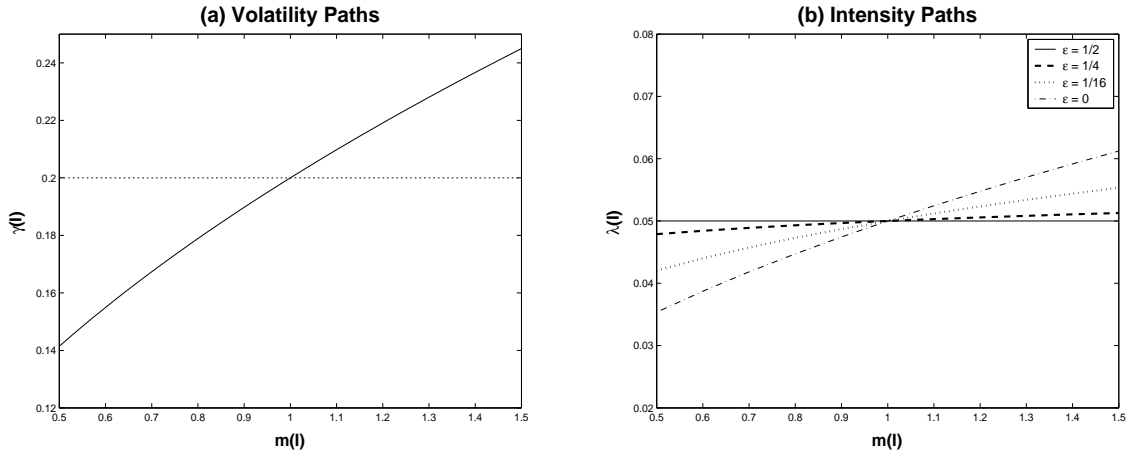


Figure 5.2: (a)  $\gamma(I)$  for different levels of  $m(I)$  (b)  $\lambda(I)$  for different levels of  $m(I)$

### 5.4.2 Default Intensity

In terms of Assumption 9 both the market index,  $I$ , and the systematic risk factor,  $\epsilon$ , are the basic ingredients to capture risk in the default intensity. Further, our moneyiness ratio,  $m(I)$ , is purely a function of the market index. We make this definition precise by specifying the default intensity as:

$$\lambda(I, \epsilon) = \bar{\lambda}[m(I)]^\epsilon = \frac{\bar{\lambda}}{\bar{\gamma}}[m(I)]^{\epsilon - \frac{1}{2}}\gamma(I) \quad \text{for } \bar{\lambda} \in \mathbb{R}_+ \quad \text{and } \epsilon \in [0, 1] \quad (5.19)$$

Notice, that with this specification the default intensity is defined as a function of the market index or as a function of the index volatility. In fact, one can assert that an accurate definition of the default intensity should be purely a function of a risk measure, that is, the volatility.

An illustration of expression (5.19) is shown in Figure 5.2(a) and Figure 5.2(b) for possible paths for (i) the index volatility for different levels of  $m(I)$  versus naive constant volatility  $\bar{\gamma} = 0.2$  and (ii) the default intensity for different levels of  $m(I)$  and different  $\epsilon = 0, \frac{1}{16}, \frac{1}{4}, \frac{1}{2}$  and constant intensity  $\bar{\lambda} = 0.05$ .

### 5.4.3 Loss Quota Distribution

In credit risk applications the loss quota is often defined to be of *beta* distribution. This is a rational assumption since the density function of a beta distribution is bounded in the interval  $[0, 1]$ . The class of beta distributions is generated by the parameters specified in the beta function and as such allows the researcher much flexibility in specifying the loss function. According to Gupton and Stein (2002) [59] rating agencies often use the beta distribution to model the recovery rate. With this specification the density is expressed as  $f(x) = \frac{1}{B(a,b)}x^{a-1}(1-x)^{b-1}I_{(0,1)}(x)$  where  $a$  and  $b$  are non-negative constants

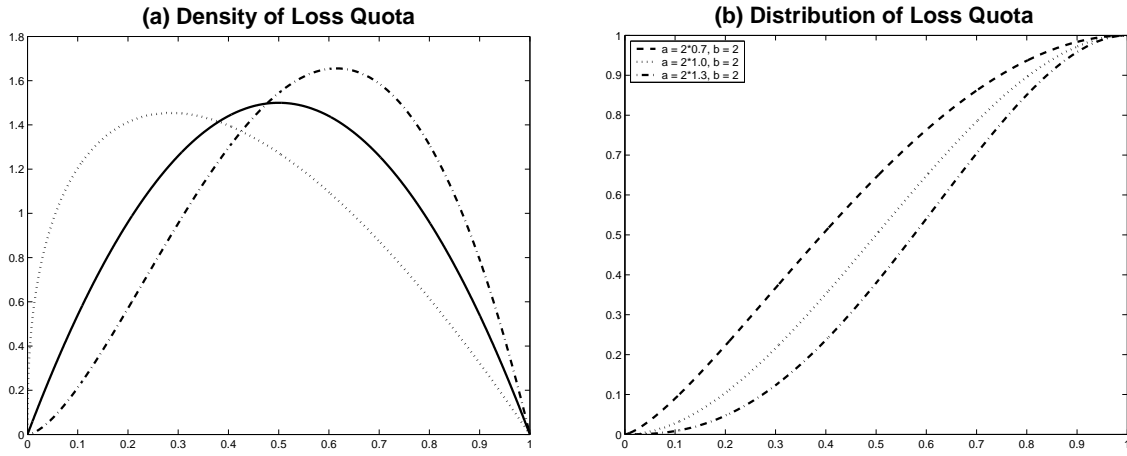


Figure 5.3: (a) Density Function of loss quota for  $m = 1.3$ ,  $m = 1$ ,  $m = 0.7$  (b) Distribution Function of loss quota for  $m = 1.3$ ,  $m = 1$ ,  $m = 0.7$

and  $B(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx$  is the beta function. Heuristically, the parameters  $a$  and  $b$  characterize the shape of the distribution with  $a$  associated with large loss distributions and  $b$  associated with small loss distributions.

Figure 5.3(a) shows us the loss quota density and Figure 5.3(b) its cumulative distribution function for three different values of the market index  $m(I)$ :  $m = 0.7$  representing a bull market,  $m = 1$  for the case where the market is at its long-run level, and  $m = 1.3$  representing a bear market. In general, we allow the loss distribution to depend on time and state. In our specification of a beta distribution this means that  $a$  and  $b$  are time and state dependent. In particular, we choose the loss quota,  $q \sim \text{Beta}(2m(I), 2)$  i.e.  $a = 2m(I)$  and  $b = 2$ , with  $a$  dependent on the market index  $m(I)$  and  $b$  is a constant. This is consistent with the preferred properties stated in assumption (2.10). We can express the loss distribution function as:

$$\tilde{K}(q, I) = \frac{1}{B(2m(I), 2)} \int_0^q x^{2m(I)-1}(1-x)dx$$

With this specification of the beta distribution we can state the following two properties for the expected loss function:

$$q^e = \mathbb{E}[q(I)] = \frac{m(I)}{1 + m(I)}$$

$$\frac{\partial q^e(I)}{\partial I} = \frac{\frac{\partial m(I)}{\partial I}}{(1 + m(I))^2} < 0$$

Then for a direct substitution for our choice of  $m(I)$  we can interpret the above properties at default as (i) at the long-run level,  $m(I) = 1$ , the loss quota value is  $\frac{1}{2}$ , (ii) at a relatively higher index level asset prices are generally higher,  $m(I) < 1$ , an investor

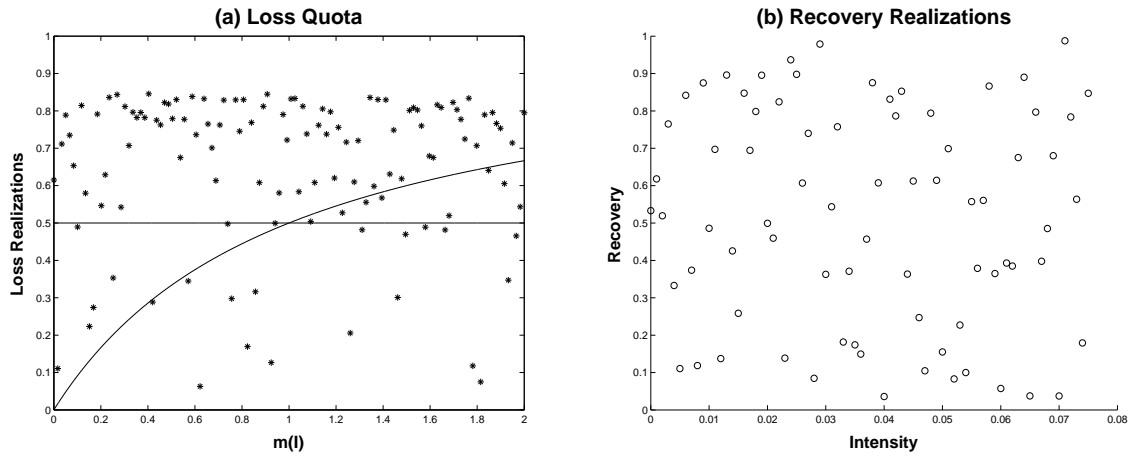


Figure 5.4: (a) Loss quota possible realizations and expected value for different values of 'moneyness'  $m$  (b) Scatter plot of Intensity versus possible Recovery Realization for different values of  $m$

expects a higher recovery rate with the expected loss quota decreasing and (iii) at a relatively lower index level asset prices are generally lower,  $m(I) < 1$ , an investor expects a lower recovery rate with the expected loss quota increasing.

Further, we show in Figure 5.4(a) the possible realizations of the loss quota (i) drawn from the beta density with the appropriate mean for each  $m$  (stars), (ii) the expected loss quota levels for different values of  $m$  (full line) in contrast with (iii) the naive approach of taking  $\bar{q} = \frac{1}{2}$  (dotted line). Finally, we illustrate a possible relation between the recovery process,  $(1 - q)$ , and the intensity,  $\lambda$ . Figure 5.4(b) shows the scatter plot of one possible recovery realization versus  $\lambda$  for different levels of the index.

## 5.5 Simulation Study

While in the past several years there has been marked progress with regard to arbitrage-free credit risk pricing models [see Jarrow and Turnbull (1995) [73], Lando (1998) [85]] they generally make the assumption of independence between the recovery rate and default intensity.

A review of recent literature reveals that many models derive the default intensity and the recovery rate as a function of several economic factors. Further, the default intensity additionally conditions on cyclical factors. On the other hand, most models, if any, lack the sophistication of capturing the empirical observation of the feedback dynamics between the default intensity and the loss quota. Empirical regularities suggest that both the default intensity and the loss quota are affected by cyclical factors. Then, if both the default intensity and the loss quota are correlated to the same macro-economic variables it is intuitive they should be modelled as correlated variables. Notwithstanding and in

accordance with Basel II (2004) [11] requirements emerging research develop models that assumes correlation between the recovery rate and default intensity. In this section we explicitly simulate the term structures for the short spread, the forward spread and the probabilities of default as impacted by the correlation between the default intensity and the loss quota.

### 5.5.1 Monte Carlo Numerical Scheme

In a complete economy setting the no-arbitrage price of a generic contingent claim can be expressed as the discounted expected value of its payoff. When in the absence of closed-form solutions Monte Carlo techniques have become a powerful computational tool in the valuation and hedging of financial securities by computing the expectations of contingent claims. Monte Carlo methods typically encompasses simulating the underlying dynamics of the price processes and the other associated risk variables that affect the price of the specified security.

In this section we aim to demonstrate the simulation framework for the term structure of credit spreads and default probabilities in a risk-neutral framework given that the price of an underlying asset  $X_t$  has stochastic dynamics and is expressed as:

$$dX_t = r(t)X_t dt + \sigma X_t dW_t \quad (5.20)$$

where  $r(t)$  is the risk-free rate,  $\sigma$  is the volatility coefficient and  $W_t$  is a standard Brownian motion. Further, we express the no-arbitrage price  $V(t)$  of a corporate bond with payoff  $f(X(t_1) \dots X(t_j))$  at time  $T = t_j$  as:

$$V(t) = e^{-r(T-t)} \mathbb{E}_t^{\mathbb{Q}}[f(X(t_1) \dots X(t_j)) | \mathcal{F}_t] \quad (5.21)$$

and the expectation is taken under the martingale measure  $\mathbb{Q}$ ,  $\mathcal{F}_t$  represents the information set of the Brownian motion  $W_t$  up to the time  $t$ , and  $T$  is the maturity of the bond.

To compute the expectation of expression (5.21), we have to simulate the risk neutral asset dynamics as shown in expression (5.20) over the time interval  $[0, T]$ . Clearly, the nonlinear SDE in (5.20) cannot be solved explicitly such that we have to resort to techniques from numerical methods. We choose the stochastic Euler scheme, hereafter Euler, which is a conditionally stable scheme, and is consistent with the Ito stochastic integral over  $[t_{i+1}, t_i]$ . In particular, we follow the Monte Carlo methodology shown in Glasserman (2003) [56] in conjunction with the Euler approximation of expression (5.20) to simulate the asset dynamics.

The Euler approximation of  $X_i$  of  $X(t_i)$  on a time grid  $0 = t_0 < \dots < t_j = T$  is given as:

$$X_{i+1} = X_i + r_i X_i [t_{i+1} - t_i] + \sigma X_i \sqrt{t_{i+1} - t_i} W_{i+1} \quad (5.22)$$

### Reference Parameters

Maturities ( $T$ )	From days up to 5 years	
Riskfree interest rate	5%	
$m(I)$	Case A: Bull Market	0.7
$m(I)$	Case B: Normal Market	1.0
$m(I)$	Case C: Bear Market	1.3
Long-run index value	$10000e^{0.5*T}$	
Fixed index volatility ( $\bar{\gamma}$ )	20%	
Fixed intensity value ( $\bar{\lambda}$ )	5%	
Fixed recovery value ( $\bar{q} = \frac{1}{2}$ )	50%	

Table 5.1: Reference values for the parameters in the model

where  $X_0 = X(0)$  is specified and  $W_i$  are standard normal random variables for  $i = 0, \dots, j-1$ . If we assume a fixed time step  $\Delta t = t_{i+1} - t_i > 0$ , we can express the Euler formula as:

$$X_{i+1} = X_i + r_i X_i \Delta t + \sigma X_i \sqrt{\Delta t} W_{i+1} \quad (5.23)$$

We apply this generic Euler formulation of a SDE to all SDE's in our Monte Carlo simulations. Finally, to setup the scheme we choose a discretization step size  $\Delta t = \frac{T}{N}$  of the time interval  $[0, T]$ ,  $N$  being a non-negative integer, and simulate  $n$  trajectories ( $j = 1 \dots n$ ) of the index process using the Euler scheme.

#### 5.5.2 Simulation Scenarios

Given our selection of the loss quota as the beta distribution, default as a rare event and the complexity of the macro-economic variables, we apply simulation techniques to generate sets of realistic scenarios for our credit spread dynamics and default probabilities. The model outputs are not just simply simulated from exogenously specified parameters but instead they are generated from scenarios of the underlying risk variables such as the loss quota, default intensity and index volatility. In our simulations we use the Monte Carlo method where the step size is 0.05 years and all our simulations concern 100 paths. In Table 5.1 we set out the reference parameters and in Table 5.2 we show all possible scenarios that our computations are based on. These tables are taken from Gaspar and Slinko (2005) [51].

#### 5.5.3 Simulation Results

From the reference parameters on the individual bonds, term structures on default-free and defaultable bonds, different systematic risk factors and the long-run index value we simulate a set of economic scenarios for short spreads and forward spreads. The functional

## Different Scenarios

Scenario	Index	Volatility	Intensity	Recovery
(1)		F	F	F
(2)		S	F	F
(3)		F	F	S
(4)		S	F	S
(5)		F	S	F
(6)		S	S	F
(7)		F	S	S
(8)		S	S	S

Table 5.2: Basic reference scenarios for simulations F=Fixed, S=Stochastic

forms for the term structure of credit spreads are described in the following proposition.

**Proposition.** Given Assumption 6, and under the martingale measure  $\mathbb{Q}$

1. The short credit spreads,  $s(t)$ , have the following functional form

$$s(t) = \lambda(t, X_t)q^e(t, X_t) > 0 \quad (5.24)$$

where

$$q^e(t, X_t) = -\int_0^1 qK(t, dq, X_t) > 0$$

can be interpreted as the locally expected loss quota (which is positive for  $q > 0$ ).

2. Then the forward credit spread  $s(t, T)$  takes the form

$$s(t, T) = \frac{\mathbb{E}_t^{\mathbb{Q}} \left[ \{r(T) + \lambda(T, X_T)q^e(T, X_T)\} e^{-\int_t^T \{r(s) + \lambda(s, X_s)q^e(s, X_s)\} ds} \right]}{\mathbb{E}_t^{\mathbb{Q}} \left[ e^{-\int_t^T \{r(s) + \lambda(s, X_s)q^e(s, X_s)\} ds} \right]} - f(t, T) \quad (5.25)$$

**Proof:** see Gaspar and Slinko (2005) [51].

In the discussion of the simulation results our comments generally apply to the bull, normal and bear market conditions else we will note the specific market condition. The spreads with zero maturity correspond to the short spread, for all other maturities correspond to the forward spread. Our simulation results are computed using the scientific

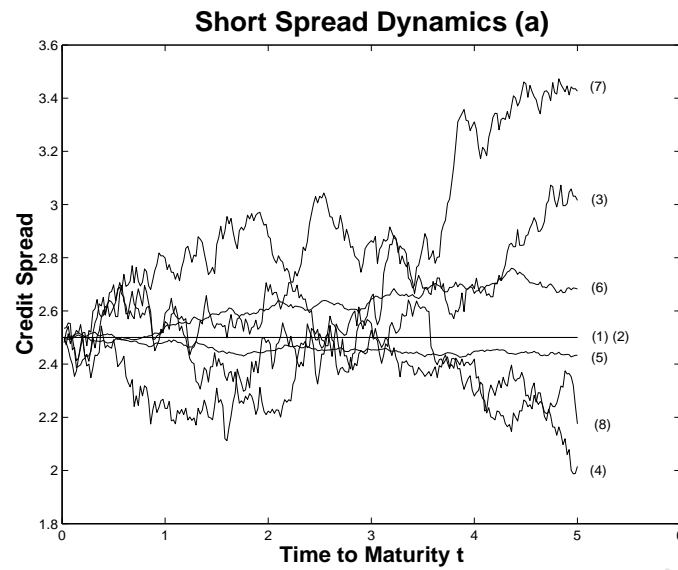


Figure 5.5: Possible Paths for the Short Spread Dynamics with Initial Index Value of  $10 * \exp(0.5 * T)$

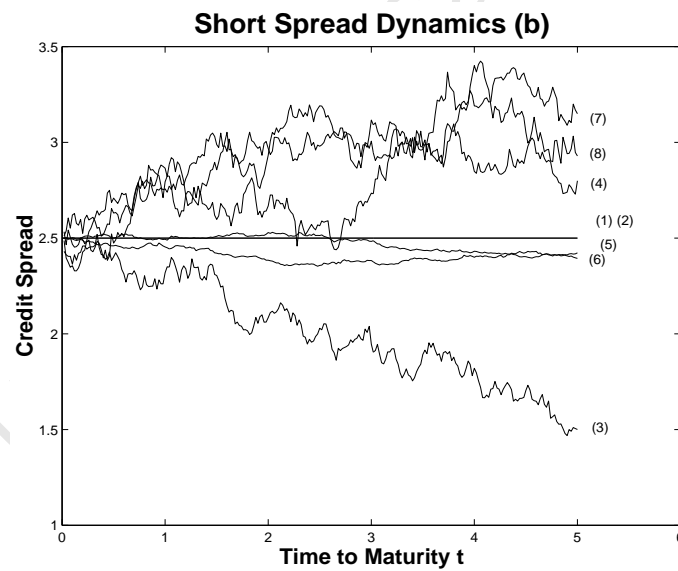


Figure 5.6: Possible Paths for the Short Spread Dynamics with Initial Index Value of  $100 * \exp(-0.01 * T)$

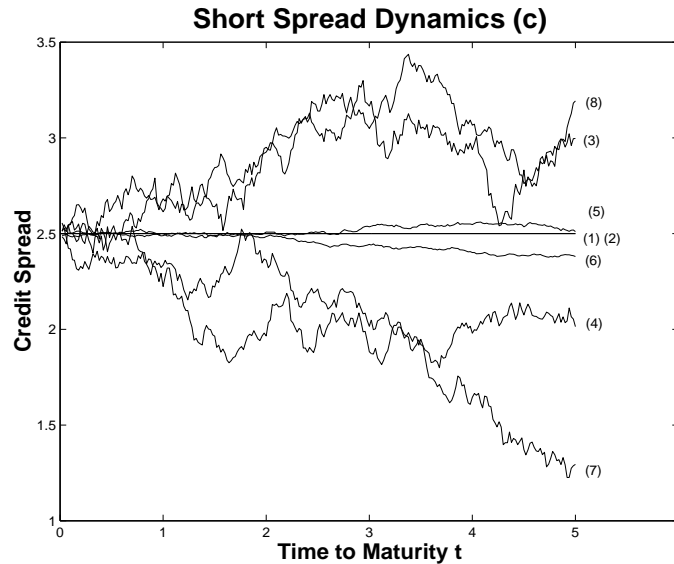


Figure 5.7: Possible Paths for the Short Spread Dynamics with Initial Index Value of  $10 * \exp(-0.5 * T)$

software package *Matlab*. The simulation code for the results is stored in the hardware accompanying this dissertation.

The short spread dynamics is given by expression 5.24. In Figures 5.5, 5.6 and 5.7 we present results for three variations for the short spread given the matrix of scenarios (see Table 5.2). These paths are arbitrary and are simulated purely for illustrative purposes. The paths are differentiated by their initial index value  $I_0$ , that is, (a)  $I_0 = 10 \exp(0.5 * T)$ , (b)  $I_0 = 100 \exp(-0.01 * T)$  and (c)  $I_0 = 10 \exp(-0.5 * T)$ . The short spreads across all paths range between 1 and 3.5 percentage points. Although the initial index value increases over time for variation (a) and decreases over time for variations (b) and (c) we observe that all paths display identical distributions and are marginally influenced by the initial index value. Gaspar and Slinko (2005) [51] argue that one factor of stochasticity in either the intensity parameter or expected loss quota show similar short spread dynamics but stochasticity in both state variables lead to higher levels of short spreads. Notwithstanding, and in contrast to their constant interest model we propose the Vasicek (1977) [118] interest model as the short rate process that implicitly gives the stochastic intensity parameter or expected loss quota an additional factor of stochasticity. As such in our setup we observe that all paths display identical characteristics and may imply that additional factors of stochasticity above two factors have a marginal impact on short spreads.

In Figures 5.8, 5.9 and 5.10 we present results for the terms structure of forward spreads for three possible market phases: (a) a bull market where the moneyiness ratio  $m = 0.7$  (b) a normal market where the moneyiness ratio  $m = 1.0$  and (c) a bear market where the moneyiness ratio  $m = 1.3$ . We notice an interesting phenomenon for both the



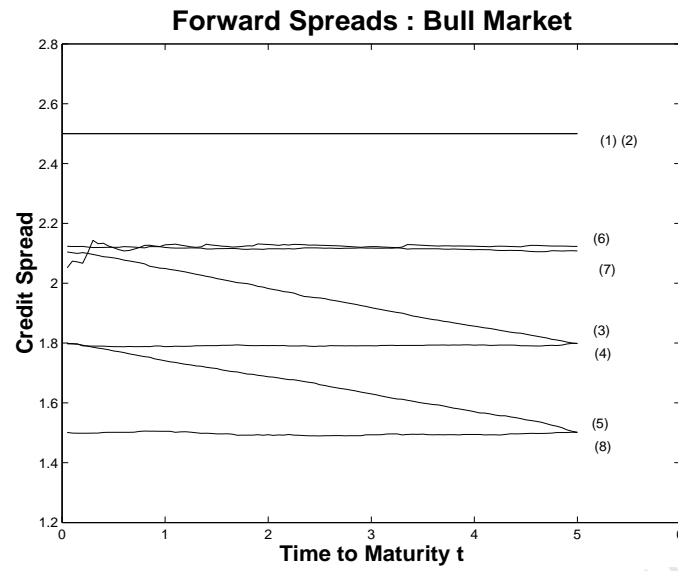


Figure 5.8: Credit Spreads for Several Maturities  $T = 0, 0.1, 0.5, 1, 1.5, 2, 3, 5$ .

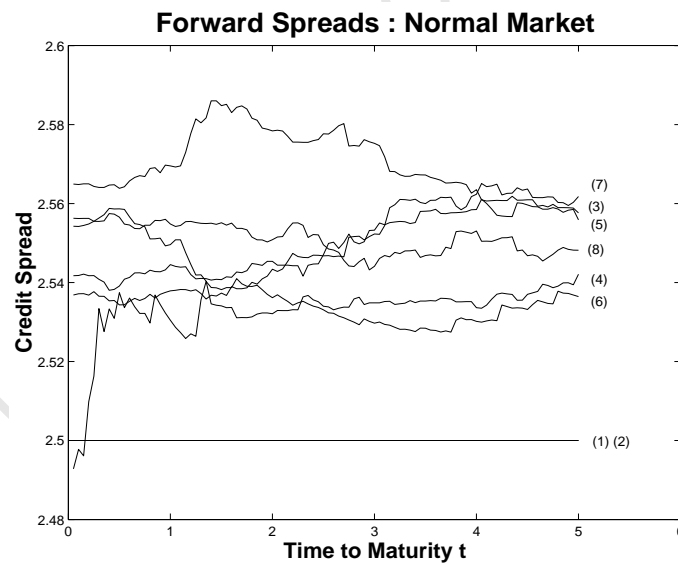


Figure 5.9: Credit Spreads for Several Maturities  $T = 0, 0.1, 0.5, 1, 1.5, 2, 3, 5$ .

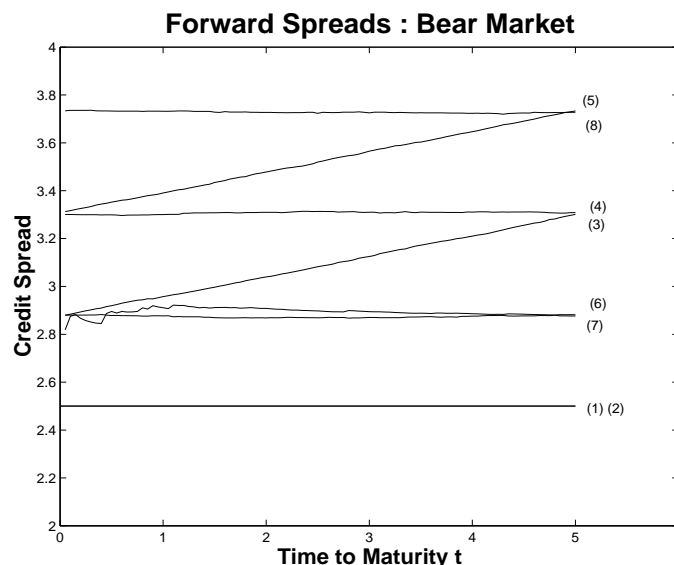


Figure 5.10: Credit Spreads for Several Maturities  $T = 0, 0.1, 0.5, 1, 1.5, 2, 3, 5$ .

bull and bear market phases. Notwithstanding, the impact of a stochastic interest rate model as an additional stochastic factor in the market index process, and its corresponding effects on the probability of default and loss given default, the forward spread term structure is relatively deterministic.

In Figure 5.8 the bull market paths with either a stochastic intensity parameter or expected loss quota show forward spreads with a downward sloping trajectory. Additionally, these paths have two factors of stochasticity in the market index process. All other paths show a flat term structure of forward spreads. The level of forward spreads are less than equal to 2.5 percent in bull markets. From our analysis of Figure 5.10 we observe that the bear market forward spread paths are an approximate image of the bull market figure, that is, where the paths are downward sloping in the bull market figure those paths are now upward sloping in the bear market figure. The level of the forward spreads are greater than equal to 2.5 percent in bear markets. Clearly, we assert that apart from the two sloping paths the trajectories of the forward spreads are generally not a naturally occurring empirical regularity. A plausible explanation for this occurrence is that, perhaps, the reference parameter values and the model specification for the intensity parameter and expected loss quota combined to give this effect. Nonetheless, our intention is to simulate the intuition underlying the model.

In contrast to both the bull and bear market forward spread paths the term structure of forward spreads for the normal market, as given in Figure 5.9, show the expected stochastic dynamics. The forward spreads generally oscillate above the 2.5 percent threshold and are in the range 2.5 to 2.6 percent. Further, we make the similar observation as for the short spread dynamics, that is, a second factor of stochasticity in either the in-

tensity process or expected loss quota have the same order of magnitude impact as one factor of stochasticity in any of the economic scenarios. Consequently, we assert that since the market index process implicitly has two factors of stochasticity any additional levels of stochasticity has negligible effect on the forward spreads. This means that in our setup and in conjunction with the specification of the state variables, that is, the intensity process and expected loss quota, the simulation results for the forward spreads are generally invariant of the matrix of scenarios but rather contingent on the selection of the market index volatility.

In Table 5.3 (see Appendix 5) we show typical simulation values of credit spreads for maturities  $T = 0, 0.1, 0.5, 1, 1.5, 2, 3, 5$  years for the matrix of scenarios in the bull, normal and bear market phases. For maturity  $T = 0$  the credit spread is the short spread and for all other maturities the values are for the forward spread.

Notice, that for the naive scenarios (1) and (2) where the intensity parameter and expected loss quota are assigned constant values we get the expected results of constant spreads across all maturities and for each market phase. Moreover, the term structure of forward spreads for short maturities are bounded away from zero against the horizon  $T$ . In contrast to structural models where the predictability of defaults imply zero short spreads this feature of intensity models is consistent with market observations.

#### 5.5.4 Prices and Survival of Credit Risky Bonds

We follow chapter 4 of this dissertation and state the value of credit risky bonds, in particular, zero-coupon bonds. The price of a credit risky zero-coupon bond  $\bar{p}(t, T)$  at time  $t$  with maturity  $T$  under zero-recovery is expressed as:

$$\bar{p}(t, T) = \mathbb{E}_t^{\mathbb{Q}} \left[ e^{-\int_t^T r(u) + \lambda(u) du} \right] \quad (5.26)$$

where the bond par value is one unit of currency is discounted with a risk-adjusted short-rate  $r_t + \lambda_t$ . The zero-recovery value of a credit risky bond is a special case of the *Recovery of Market Value* (RMV) model of Duffie and Singleton (1999) [42]. In the RMV model the default payoff is specified as a fraction,  $(1 - q^e)$ , of the pre-default value of the credit risky bond and with  $\lambda$  given as the intensity parameter. Then the price of a zero-coupon credit risky bond at time  $t$  with maturity  $T$  under non-zero recovery is expressed as:

$$\bar{p}^{RMV}(t, T) = \mathbb{E}_t^{\mathbb{Q}} \left[ e^{-\int_t^T r(u) + \lambda(u) q^e du} \right]$$

Further, under the RMV model we find the implied survival probabilities to be:

$$P(t, T) = \mathbb{E}_t^{\mathbb{Q}} \left[ e^{-\int_t^T r(u) + \lambda(u) q^e du} \right] \quad (5.27)$$

Typical values for prices of defaultable zero-coupon bonds with recovery for several maturities  $T = 0, 0.1, 0.5, 1, 1.5, 2, 3, 5$  years are given in Table 5.4 (see Appendix 5). These quantities are simulated under the various economic scenarios (see Table 5.2) for

bull, normal and bear market conditions.

For naive scenarios (1) and (2) where the intensity process or the expected loss quota is only dependent on the market index volatility we observe that the prices of bonds across all maturities is relatively constant under all market phases. In addition, for all other scenarios and maturities bond prices are stochastic. Further, the prices at high maturities ( $T = 5$ ) are relatively low and constant. This implies that investors are clearly averse to corporate bonds with longer maturity horizons. While scenarios (3) and (4) depend on the intensity parameter, (5) and (6) on the expected loss quota and (7) and (8) on both state variables, bond prices increase from higher to lower maturity horizons albeit with one order of magnitude jump from  $T = 5$  to  $T = 0.1$ . This is consistent with empirical regularities that credit risky bond prices tend to their par value as the time to maturity approaches zero. In the event of no default the bond price should equal its par value at time  $T = 0$ . Notwithstanding, from the numerical results we notice that at maturity bond prices significantly close to but not equal to their par value. This underpricing is attributed to being a model deficiency. Nonetheless, the underpricing improves as we move from a bear to a bull market in the business cycle. The trend for the underpricing across the matrix of economic scenarios is that it decreases for a bull market, is mixed for a normal market and increases for a bear market. This is consistent with our hypothesis on the effects of macro-economic variables and the correlation of the probability of default and expected loss quota in intensity models.

In Table 5.5 (see Appendix 5) we show values for prices of zero-recovery defaultable zero-coupon bonds for several maturities  $T = 0, 0.1, 0.5, 1, 1.5, 2, 3, 5$  years and in accordance with expression (5.27). In our model setup this means that with the expected loss quota switched off, the matrix of economic scenarios and maturities comprise three variations, that is, constant intensity, intensity as a function of a constant volatility and intensity as a function of stochastic volatility. For the constant intensity scenarios (1), (2), (3), (4) the bond prices are low and relatively constant across all maturities. For the intensity with constant volatility, scenarios (5) and (7), and intensity with stochastic volatility, scenarios (6) and (8), bond prices appreciate from high maturities ( $T = 5$ ) to low maturities ( $T = 0.1$ ) with a marked increase in prices from  $T = 0.1$  to maturity  $T = 0$ . The prices of bonds with zero-recovery generally reflect the trend of the results for prices with recovery (see Table 5.4, Appendix 5) albeit with lower values. This is consistent with intuition that bonds with zero-recovery are cheaper than bonds with recovery.

In conjunction with zero-recovery credit risky bond prices we show the implied survival probabilities of zero-coupon bond prices for several maturities  $T = 0, 0.1, 0.5, 1, 1.5, 2, 3, 5$  years in Table 5.6 (see Appendix 5). An inspection of the results show that the implied survival probabilities reflect the trend underlying the prices of zero-recovery bonds. More specifically, for the constant intensity scenarios (1), (2), (3) and (4) survival probabilities are low and relatively constant across all maturities. This is consistent with investors risk preferences of being averse to investing in securities of high probability of default. For the scenarios where the market index impacts the intensity process the survival probabilities

increase from high maturities ( $T = 5$ ) to low maturities ( $T = 1$ ). Intuitively investors believe that credit risky bonds with longer horizons to maturity are deemed to be more risky than bonds with shorter terms to maturity. Generally, securities in a bull market have a marginally higher likelihood of survival than securities in a bear market.

### 5.5.5 Sensitivity Effects on Price and Survival of Credit Risky Bonds

Recall that we earlier specified the intensity process as a function of the parameter  $\epsilon$  in

$$\lambda(I, \epsilon) = \bar{\lambda}[m(I)]^\epsilon$$

where  $\epsilon$  is a variable that measures sensitivity of a corporate's default risk relative to market dynamics. The underlying intuition of the sensitivity measure,  $\epsilon$ , is that a strong balance sheet and long-term profitable order books serve as a proxy for low credit risk corporates while a weak balance sheet and short-term income generating order books proxies higher credit risk corporates. As a consequence, the probability of default of corporates with low credit risk is less affected by the impact of the systematic risks on its activities relative to corporates with higher credit risk that demonstrate an increased sensitivity to systematic risks.

Following Gaspar and Slinko (2005) [51] we consider some of our previous simulation variations for prices and survival of credit risky bonds under the additional constraint of market sensitivity. In particular, we perform simulations under three different values of  $\epsilon$ : high  $\epsilon = 1/2$ , medium  $\epsilon = 1/4$  and low  $\epsilon = 1/16$  and for several maturities  $T = 0, 0.1, 0.5, 1, 1.5, 2, 3, 5$  years. Moreover, the simulation results are for a high  $\epsilon$  bull market, a medium  $\epsilon$  normal market and a low  $\epsilon$  bear market.

In Figures 5.11, 5.12 and 5.13 and Table 5.7 (see Appendix 5) we show the simulation results for a high  $\epsilon$  bull market, a medium  $\epsilon$  normal market and a low  $\epsilon$  bear market. We notice that in a high  $\epsilon$  bull market with the economic scenarios (5), (6), (7) and (8) under a stochastic intensity specification the term structure of forward spreads are marginally higher than the standard ( $\epsilon = 1$ ) bull market forward spreads (see Table 5.3, Appendix 5). For a medium  $\epsilon$  normal market the term structure of credit spreads are generally identical to the standard normal market credit spreads (see Table 5.3, Appendix 5). Finally, for a low  $\epsilon$  bear market with economic scenarios (5), (6), (7) and (8) under stochastic intensity specification the term structure of credit spreads are marginally lower than the standard ( $\epsilon = 1$ ) bear market credit spreads (see Table 5.3, Appendix 5).

The value for the simulation results given in in Figures 5.11, 5.12 and 5.13 are shown in Table 5.8 (see Appendix 5) for prices of zero-coupon bonds with recovery, in Table 5.9 (see Appendix 5) for prices of zero-coupon bonds with zero-recovery and in Table 5.10 (see Appendix 5) for implied survival probabilities of zero-coupon bonds with zero-recovery. A comparison of these results with the standard results, that is,  $\epsilon = 1$  in Tables 5.4, 5.5 and 5.6 (see Appendix 5), show that the values for scenarios (7) and (8) demonstrate

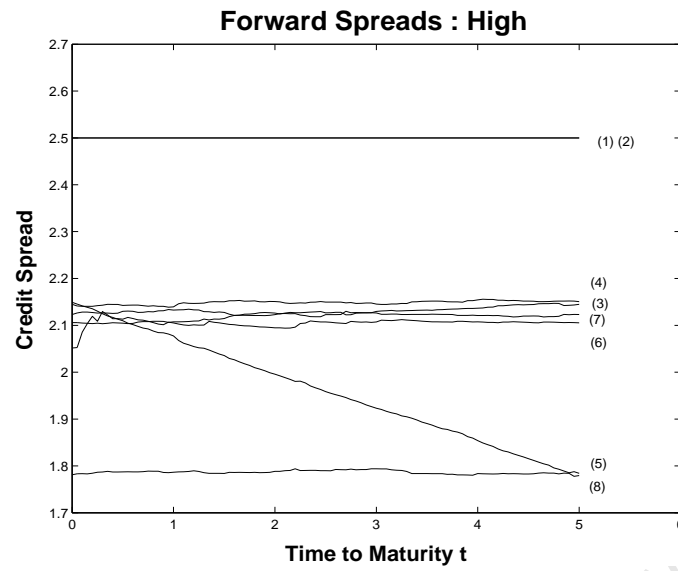


Figure 5.11: Credit Spreads for Several Maturities at High  $\epsilon$ .

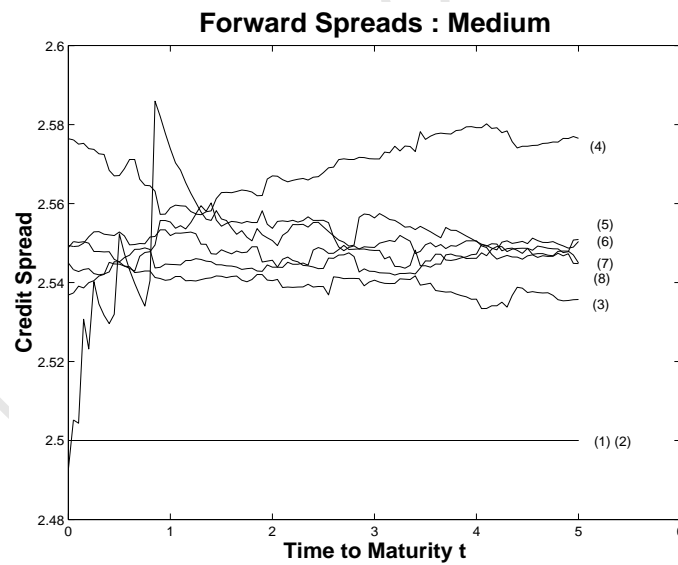


Figure 5.12: Credit Spreads for Several Maturities at Medium  $\epsilon$ .

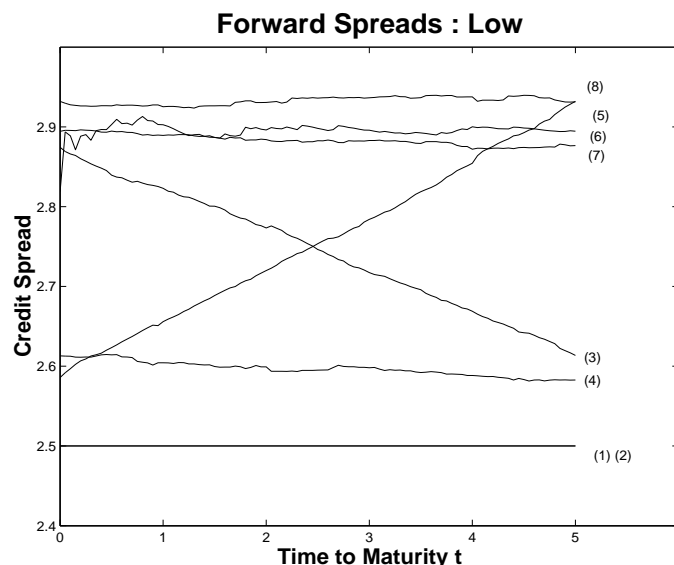


Figure 5.13: Credit Spreads for Several Maturities at Low  $\epsilon$ .

consistent differences. More precisely, for the prices of zero-coupon bonds with recovery we notice that the high  $\epsilon$  bull market and medium  $\epsilon$  normal market show lower values while the low  $\epsilon$  bear market show higher values. As a result, and expressly observed in the bear market, the model perfectly captures the market practice that investors reward corporates that are deemed to be of high credit worthiness. However, for the prices of zero-coupon bonds with zero-recovery we notice that the high  $\epsilon$  bull market show lower values while both the medium  $\epsilon$  normal market and low  $\epsilon$  bear market show higher values. Even under zero-recovery the medium  $\epsilon$  normal market conditions underlines the proposition that investors tend to reward corporates based on a perceived sensitivity to default on its obligations. A poignant observation is that for both economic scenarios (7) and (8) the intensity parameter and the expected loss quota has a stochastic dependence on the market index whereas the other scenarios has at most a single stochastic state variable dependent on the market index. Finally, the implied survival probabilities exactly reflects the results for zero-coupon bond prices with zero-recovery, that is, higher bond prices translates into higher probabilities of survival.

### 5.5.6 Credit Spreads at Higher Maturities

In this section we make a natural extension to our base case forward spread scenarios by increasing the maturity of the term structure of credit spreads from 5 to 20 years and analyze our results accordingly. Figures 5.14, 5.15 and 5.16 gives us the term structures of credit spreads for the bull, normal and bear market phases and the corresponding simulation results are shown in Table 5.11 (see Appendix 5). Notably, the results for the higher maturity credit spreads are quite similar to that for the base case scenarios. Through empirical regularities we know that credit spreads should be wider with the

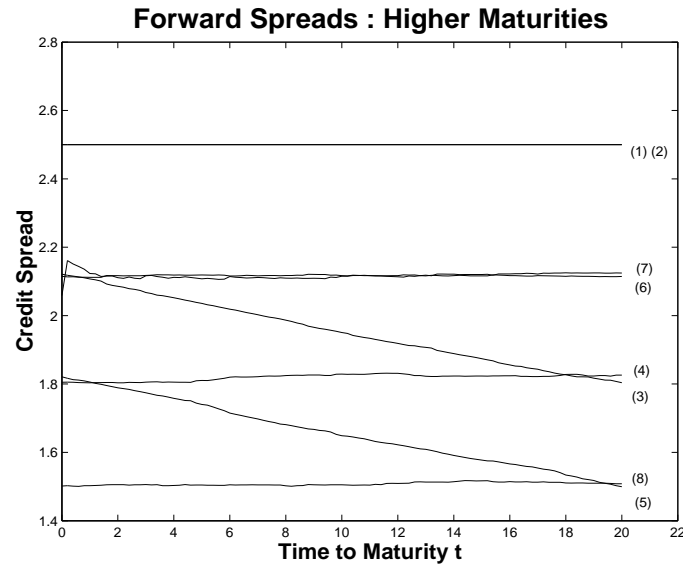


Figure 5.14: Credit Spreads for a Bull Market at Several Higher Maturities  $T = 5, 6, 7, 8, 10, 12, 15, 20$ .

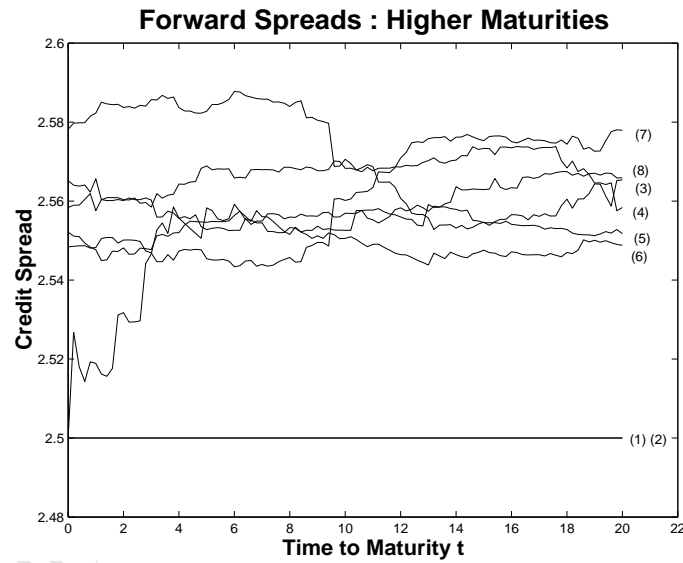


Figure 5.15: Credit Spreads for a Normal Market at Several Higher Maturities  $T = 5, 6, 7, 8, 10, 12, 15, 20$ .

increase in terms to maturity. Clearly, our results do not reflects this market occurrence. This implies that the model may not be suitable for modelling term structures at higher maturities.



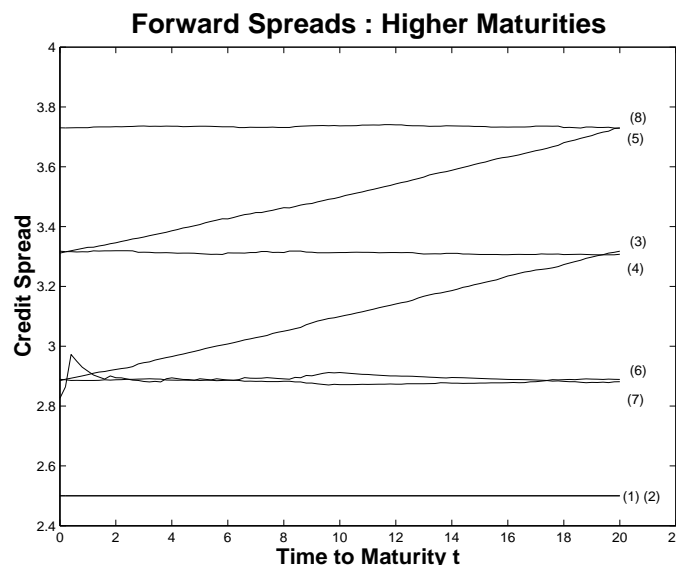


Figure 5.16: Credit Spreads for a Bear Market at Several Higher Maturities  $T = 5, 6, 7, 8, 10, 12, 15, 20$ .

## 5.6 Macro-economic Factors and Credit Spreads

A review of literature shows that several studies attempt to explain the determinants of credit spreads. We give a concise overview of the evidence and motivation concerning taxes and liquidity that are deemed to affect credit spread changes.

Generally, a large portion of credit risk empirical studies found in literature is based on corporate bond data from the US market and as such the reflected credit spreads will encompass features prevalent in this market. Specifically, Treasury bonds are tax exempt at state level whereas corporate bonds are taxable instruments. Investors select their investments across asset classes based on after tax expected returns. This means that by the no-arbitrage assumption the yield on corporate debt will capture the tax effect to compensate the investor's tax liability. In their study Elton et al. (2005) [45] show that with a benchmark tax rate of 4.875 percent the tax component can comprise between 35-75 percent of credit spreads across rating and maturity. On the other hand, one may argue that the tax regime in certain jurisdictions may exempt corporate bond investors from any tax liability and as there would be no tax impact on credit spreads.

A quick cross-section of the literature show various insights into the modelling of credit risk. Of particular interest is the relationship between credit spreads and the business cycle. While this is a frontier strand of literature research by Amato et al. (2006) [7] demonstrate the empirical relationship between the term structure of the credit spreads and the macro-economy. Based on their review of literature they argue that both empirical and theoretical studies motivate real economic activity and inflation as determinants of credit spreads. More specifically, for their study they define and estimate a multi-factor

affine term structure model for Treasury yields and for corporate spreads they define a doubly-stochastic intensity based model. They implement both the Treasury yield and corporate spread model with the same set of macro-economic variables and latent factors. Specifically, the macro variables are defined as real activity and inflation, respectively. Additionally, corporate spreads are assumed to be affected by a latent *corporate* factor. The study results are primarily based on BBB-rated industrial entities but is also extended to evaluate spreads for banks and speculative grade industrial entities.

This study makes use of data on corporate bond yields and extracted from Bloomberg's Fair market Value yield curves. Corporate spreads for BBB-industrials is constructed from a sample that spans from April 1991 to April 2004. The fact that this is a period that encompasses more than one business cycle makes the explanations of the macro variables affecting corporate spreads more plausible. Summary statistics shows that the unconditional means of BBB spreads are increasing in maturity and the term structure of unconditional volatility is upward sloping. The spreads are positively skewed and this effect is particularly significant at long maturities. In addition, there is some evidence to show that the spreads exhibit a non-Gaussian distribution. Moreover, the results show that the spreads are highly correlated across maturities with approximately 99 percent in the variation captured by the first three principal components. From the plots of the macro-economic factors it can be seen that the dynamics of the factors appear to be affected by business cycle frequencies. In particular, real activity displays characteristic behavior by increasing after a recession and markedly falling prior to a recession. Further, inspection of the correlation statistics shows that both real activity and inflation are correlated with Treasury yields at all maturities with real activity showing relatively higher correlations than inflation for the sample time series. In contrast, real activity is negatively correlated with spreads while the correlation between inflation and spreads is approximately zero.

## 5.7 Macro-economic Factors and Probability of Default

In this section we give concise comments on macro-economic factors and their impact on credit risk through the default probability. In particular, we consider the studies by Bruche and González-Aguado (2006) [24] and Couderc and Renault (2005) [28] and reflect on the results of their examination of an array of common determinants that affect default probability changes through the business cycle.

In their research Bruche and González-Aguado (2006) [24] propose a model that suggests that the correlation of recovery rates and default probabilities is a consequence of an unobserved credit cycle as opposed to solely business cycle effects. Their model uses input data from the Altman-NYU Salomon Center Corporate Bond Default Master Database to estimate credit downturns which they compare to recession periods as published by the National Bureau of Economic Research (NBER). Their model predicts credit downturns that precedes the onset of recessions and typically continues until after the end of a recession. Their estimated credit cycle indicator captures mild correlation with several

macro-economic factors, for example, a correlation of 34 percent with GDP growth and 36 percent with the S&P 500 return. They assert that this shows evidence that the credit cycle is associated with but separate from the business cycle. Further, they argue that this is a plausible hypothesis as to why previous research, for example, Altman et al. (2003) [1], show that macro-economic factors only partially accounts for the variation of recovery rates. Finally, this study shows that while considering the dependence between recovery rates and probabilities of default it is prudent to additionally consider the dynamic nature of credit risk to capture a fairer approximation of the underlying credit risk.

The research by Couderc and Renault (2005) [28] investigates the determinants of default probability changes of individual corporates. In particular, they analyze and quantify the sensitivity of the default intensity to changes in financial markets, business cycle and credit indicators, and examine their persistency. For sources of financial markets information they use the stock and bond markets and subsequently test the following factors on default intensities.

**Annual Return on S&P 500:** The asset value is a function of the market index with the implication that a higher return on the index shifts the corporate further away from default. The corporate's leverage is negatively correlated to its equity value with the effect that increases in equity prices lowers the probability of default. In addition, the short and medium term economic outlook is generally priced into the market index returns. As a result we expect a positive correlation between economic growth and index returns. Consequently, this has a negative impact on default intensities.

**Volatility of S&P 500 returns:** In a classic firm value model the primary drivers' of the default dynamics are the asset volatility and leverage. The asset volatility is generally proxied by the volatility of the equity returns. A rational expectation is for the volatility to have a positive impact on the default intensities.

**10 Year Treasury Yield:** The implication of higher interest rates is increased costs of debt financing. Hence a rational expectation would be that this variable would positively influence the probability of default. An empirical observation is that interest rates tend to be lower in periods of lower economic growth rates and higher in periods of higher economic growth rates. Thus the true impact of interest rates on the default intensity tends to be ambiguous and may depend on the strength of the obligor's balance sheet.

**Slope of the Term Structure:** Generally we associate upward sloping term structure of interest rates with robust economic growth outlook. Notwithstanding, this could also simultaneously reflect higher forward rates. As such this variable is expected to have a negative impact on medium to long-term default intensities.

To further quantify and assess the impact of systematic factors on default intensities Couderc and Renault (2005) [28] suggest that it is important to analyze information from the business cycle. They argue that it is cogent to consider business cycle indicators since

the return on the market index does not fully capture the dynamics of the state of the economy. These variables are described below.

**Real GDP Growth:** is considered as an indicator of the current macro-economic state and this variable is likely to be negatively correlated with the near term default intensity.

**Industrial Production Growth:** is considered an alternate growth indicator and its effect should be identical to that of real GDP growth. This indicator has the additional advantage of being updated more regularly.

**Personal Income Growth:** This indicator is considered to reflect an identical effect to both the real GDP and industrial production growth variables. Additionally, this indicator is characteristically volatile and should demonstrably be less persistent. Moreover, personal income growth implicitly transmits lagged economic information, for example, robust business growth will be reflected in the near term dynamics of this variable.

**CPI Growth:** Importantly, inflation is an indicator of economic dynamics. Since high inflation associated with increasing growth the expectation is that inflation will be negatively correlated with short-term default intensities.

It is intuitive to consider specific credit factors when analyzing the systematic dynamics on default intensities. In particular, credit spreads has been identified as a crucial indicator on both movements of underlying default intensities and future information on default intensities. Further, if variations in the default risk premium affects the credit spread then the spread should capture changes in the expectations of the default state of the economy. Hence spread indicators should be dominantly persistent than other market factors. We discuss credit spreads and their informational content on default probabilities below.

**Spread of Long-Term BBB Bonds over AAA Bonds:** As both AAA and BBB bonds are regarded as defaultable assets then this indicator captures the risk aversion of investors and may act as a proxy for their risk forecast. In addition, if there are mixed effects priced into the BBB spread then this variable extracts out those effects.

**Net Issues of Treasury Securities:** Generally, the proceeds from the issue of Treasury bonds is used to finance short-term government spending. Also, it is a measure of a higher deficit and economic stress and thus should increase short-term default intensities. In addition, increased government borrowing may result in fewer opportunities for private entities to issue debt with a consequent increase in financial stress for corporates. On the other hand, if the issue of Treasury bonds is used for investment projects then this implies long-term economic growth and a corresponding decrease in default intensities.

**Influence of Financial Markets:** Based on their model Couderc and Renault (2005) [28] show results that are consistent with previous findings on credit spreads to the extent

they show evidence that financial markets impact default intensities. In particular, gains in the market index lowers the probability of default whereas higher probabilities are associated with increases in volatility. Further, they find the slope of the term structure of interest rates is significant and has a decided impact on default intensities. Decreasing short-term yields results in increased default intensities that corresponds with the caveat that lower interest rates are related with recessions. A sharp recent slope of the term structure of interest rates correlates with higher default intensities. Bonds with a short-term maturity are sensitive to contemporaneous slope changes to the extent that increasing slopes correlates with higher intensities and spreads.

They claim that while previous studies on credit spreads imply that the effects of the market index has a stronger impact than interest rates on default intensities their results show plausible evidence to contrast this effect. For investment grade bonds the effects of a decreased long-term yield in conjunction with an increase in the slope of the term structure of interest rates has a similar impact on intensities relative to a decrease of the S&P 500 return.

**Business Cycle Effects:** From their study Couderc and Renault (2005) [28] suggest that the business cycle has a strong impact on the default cycle. Consistent with previous research business growth tends to lower default intensities. They observe that personal income growth (PIG) has stronger impacts than real GDP growth. However, GDP growth impacts tend to be more persistent but effects of PIG on the dynamics in default intensities is ambiguous. A check on likelihood ratios, they infer that lower default intensities imply forward increases PIG. Both PIG and CPI growth have a vanishing impact with decreasing credit quality. Clearly, business cycle information allows the researcher to contrast its effects relative to financial markets information and consistent with previous studies their results show that business cycle information is more efficient than financial markets information.

**Credit Markets Effects:** By inspecting their model results Couderc and Renault (2005) [28] assert that credit information embeds strong explanatory power and is especially implicit in the BBB spread. The credit information impacts appear to be identical across all credit qualities. For contemporaneous changes in net Treasury issues they observe slight but statistically significant effects on default intensities and suggest that the underlying reason could be attributed to its weak near term information value. Further, they find evidence that both the BBB and investment grade spreads embeds minor information on default intensities.

## 5.8 Conclusion

In this chapter we adapted the study by Gaspar and Slinko (2005) [51] to show an intensity-based reduced form model where the probability of default and the loss given default are correlated and dependent on a stochastic macro-economic index. In particular,

we illustrate the empirical observation that show both the default intensity and recovery rates are sensitive to macro-economic effects.

Notwithstanding the significant advances in the pricing of credit risk over the past several years macro-economic influences still has a marginal role in most pricing models. In our numerical example we assert that the treatment of macro-economic factors improves the pricing of credit risk term structures. Given this context we calibrate the model to simulate business cycle effects by constructing periods of increased credit risk in economic downturns and decreasing credit risk in bull markets. A potential advantage of the macro-economic impact on pricing credit risk is that the level of risk can be identified earlier in the business cycle and as such acts as an early warning system for management and investors.

Notably the study by Altman et al. (2003) [1] postulates that the market for defaulted debt may be of rigid capacity to the extent that it violates the standard asset pricing assumption of perfectly elastic markets. As a consequence, in periods of high default rates the market is flexible to absorb all defaulted debt with the result of depressed prices for these securities. Additionally, if we consider the standard modelling assumption of the recovery rate being specified as a proportion of par then in periods with high density defaults the recovery rate will clearly be lower. While this makes for economic sense they find that the expected statistical effect of macro-economic variables on recovery rates is less significant.

## 5.9 Appendix 5

**Theorem 1. (Existence of Intensity)** Assume that  $\nu$  admits an intensity and define  $\nu(t, dq, X_t) = M_t(dq, X_t)dt$ ,  $\mathbb{Q}$ -a.s. where  $M_t(dq, x)$  is a deterministic measure on  $E$  for any fixed  $x$  and  $t$ .

Let  $\hat{\nu}(dt, dq) = m_t(dq)dt$  be a deterministic compensator for some Marked Poisson Process  $\hat{\mu}$ .

Assume that:

- (i)  $M(t, dq, x)$  is measurable w.r.t  $\mathcal{G}^W$
- (ii)  $M(t, dq, x)$  is absolutely continuous w.r.t  $m(t, dq)$  on  $\varepsilon$ , that is,

$$M_t(dq, x) \ll m_t(dq)$$

Then there exists a  $\mathcal{G}_t^W$ -DSMPP  $\mu$ , such that its compensator is of the form (5.10).

**Proof:** see Gaspar and Slinko (2005) [51].

**Proposition 2.** Consider a  $T$ -defaultable claim  $X$ . For the purpose of computing expectations, and in particular its price at time  $t \leq T$

$$\mathbb{E}_t^{\mathbb{Q}} \left[ e^{-\int_t^T r_s ds} V(T) X \middle| \mathcal{F}_t \right]$$

it is equivalent to use the following two dynamics for the remaining value process

$$\begin{aligned} \frac{dV(t)}{V(t-)} &= - \int_0^1 q\mu(dt, dq) \\ V(t) &= v \end{aligned} \tag{5.28}$$

$$\begin{aligned} \frac{dV(t)}{V(t-)} &= -q^e(t-, X_{t-})dN_t \\ V(t) &= v \end{aligned} \tag{5.29}$$

where  $\mu$  is a DSMPP with compensator  $\nu(t, X_t) = \lambda(t, X_t)K(t, dq, X_t)dt$ ,  $N$  is a Cox process with intensity  $\lambda(t, X_t)$  and we define

$$q^e(t, X_t) = \int_0^1 K(t, dq, X_t)$$

**Proof:** see Gaspar and Slinko (2005) [51].

**Case A:Bull Market**

$T$	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
0	2.5000	2.0517	2.1163	2.1137	1.8167	1.7895	1.4821	
0.1	2.5000	2.0759	2.1166	2.1076	1.8175	1.7826	1.4836	
0.5	2.5000	2.1230	2.1171	2.0830	1.8210	1.7595	1.4822	
1	2.5000	2.1056	2.1145	2.0531	1.8141	1.7278	1.4802	
1.5	2.5000	2.1173	2.1139	2.0258	1.8122	1.7007	1.4825	
2	2.5000	2.1086	2.1154	1.9926	1.8103	1.6773	1.4854	
3	2.5000	2.1089	2.1200	1.9275	1.8125	1.6147	1.4865	
5	2.5000	2.1109	2.1137	1.8108	1.8035	1.4983	1.4896	

**Case B:Normal Market**

$T$	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
0	2.5000	2.4928	2.5622	2.5493	2.5637	2.5518	2.5562	
0.1	2.5000	2.6293	2.5616	2.5493	2.5663	2.5506	2.5566	
0.5	2.5000	2.5636	2.5605	2.5481	2.5730	2.5535	2.5628	
1	2.5000	2.5667	2.5650	2.5520	2.5743	2.5495	2.5729	
1.5	2.5000	2.5593	2.5678	2.5509	2.5757	2.5463	2.5734	
2	2.5000	2.5516	2.5661	2.5496	2.5775	2.5474	2.5694	
3	2.5000	2.5598	2.5569	2.5461	2.5653	2.5541	2.5747	
5	2.5000	2.5550	2.5452	2.5400	2.5563	2.5570	2.5668	

**Case C:Bear Market**

$T$	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
0	2.5000	2.8189	2.8765	2.8729	3.3260	3.3087	3.7405	
0.1	2.5000	2.8936	2.8751	2.8829	3.3226	3.3147	3.7406	
0.5	2.5000	2.9017	2.8761	2.9226	3.3198	3.3465	3.7352	
1	2.5000	2.8857	2.8745	2.9637	3.3194	3.3839	3.7421	
1.5	2.5000	2.8783	2.8717	3.0073	3.3131	3.4235	3.7535	
2	2.5000	2.8730	2.8694	3.0493	3.3088	3.4632	3.7527	
3	2.5000	2.8750	2.8731	3.1301	3.3098	3.5535	3.7457	
5	2.5000	2.8760	2.8730	3.3080	3.2902	3.7232	3.7451	

Table 5.3: Credit Spreads for Several Maturities  $T = 0, 0.1, 0.5, 1, 1.5, 2, 3, 5$ . For  $T = 0$  it is the short spread, for all others the forward spread. (spreads in percentage points)



**Case A: Bull Market**

$T$	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
0	0.0084	0.9056	0.9109	0.9057	0.9289	0.9247	0.9508	
0.1	0.0092	0.3017	0.3017	0.3034	0.3019	0.3094	0.3079	0.3168
0.5	0.0089	0.0820	0.0820	0.0830	0.0822	0.0844	0.0841	0.0864
1	0.0088	0.0433	0.0433	0.0435	0.0432	0.0442	0.0442	0.0452
1.5	0.0088	0.0293	0.0293	0.0294	0.0292	0.0299	0.0299	0.0307
2	0.0088	0.0222	0.0222	0.0223	0.0222	0.0226	0.0225	0.0231
3	0.0088	0.0149	0.0149	0.0149	0.0149	0.0152	0.0152	0.0155
5	0.0088	0.0090	0.0090	0.0091	0.0091	0.0092	0.0092	0.0092

**Case B: Normal Market**

$T$	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
0	0.0092	0.8909	0.895	0.8991	0.8927	0.8965	0.8995	
0.1	0.0091	0.2974	0.2986	0.3004	0.2980	0.2990	0.2998	
0.5	0.0090	0.0809	0.0814	0.082	0.0811	0.0816	0.0817	
1	0.0091	0.0424	0.0426	0.0427	0.0423	0.0426	0.0427	
1.5	0.0090	0.0288	0.0289	0.0288	0.0287	0.0288	0.0290	
2	0.0090	0.0217	0.0219	0.0218	0.0217	0.0218	0.0219	
3	0.0090	0.0147	0.0147	0.0146	0.0147	0.0146	0.0147	
5	0.0089	0.0089	0.0089	0.0089	0.0089	0.0088	0.0088	

**Case C: Bear Market**

$T$	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
0	0.0086	0.8844	0.8812	0.8750	0.8708	0.8593	0.8455	
0.1	0.0087	0.2951	0.2939	0.2914	0.2903	0.2863	0.2827	
0.5	0.0090	0.0805	0.0799	0.0793	0.0791	0.0780	0.0772	
1	0.0089	0.0422	0.0419	0.0415	0.0414	0.0407	0.0406	
1.5	0.0089	0.0285	0.0284	0.0281	0.0279	0.0275	0.0274	
2	0.0089	0.0216	0.0215	0.0212	0.0211	0.0208	0.0208	
3	0.0089	0.0144	0.0145	0.0142	0.0141	0.0140	0.0139	
5	0.0089	0.0087	0.0087	0.0086	0.0086	0.0083	0.0084	

Table 5.4: Prices of Zero-Coupon Bonds with Recovery for Several Maturities  $T = 0, 0.1, 0.5, 1, 1.5, 2, 3, 5$ . Three different market conditions: bull, normal and bear.

**Case A: Bull Market**

$T$	(1)(2)(3)(4)	(5) (7)	(6) (8)
0	0.0076	0.7994	0.8582
0.1	0.0079	0.2670	0.2861
0.5	0.0081	0.0732	0.0780
1	0.0080	0.0388	0.0409
1.5	0.0079	0.0264	0.0278
2	0.0079	0.0201	0.0210
3	0.0079	0.0137	0.0141
5	0.0079	0.0085	0.0085

**Case B: Normal Market**

$T$	(1)(2)(3)(4)	(5) (7)	(6) (8)
0	0.0077	0.7950	0.7948
0.1	0.0077	0.2648	0.2650
0.5	0.0078	0.0723	0.0734
1	0.0079	0.0380	0.0386
1.5	0.0078	0.0257	0.0261
2	0.0078	0.0194	0.0194
3	0.0078	0.0131	0.0131
5	0.0079	0.0079	0.0079

**Case C: Bear Market**

$T$	(1)(2)(3)(4)	(5) (7)	(6) (8)
0	0.0078	0.7937	0.7478
0.1	0.0076	0.2642	0.2496
0.5	0.0077	0.0718	0.0681
1	0.0077	0.0373	0.0357
1.5	0.0077	0.0251	0.0242
2	0.0078	0.0188	0.0182
3	0.0079	0.0125	0.0122
5	0.0079	0.0074	0.0074

Table 5.5: Prices of zero-recovery, Zero-Coupon Bonds for Several Maturities  $T = 0, 0.1, 0.5, 1, 1.5, 2, 3, 5$ . Three different market conditions: bull, normal and bear.

**Case A: Bull Market**

$T$	(1)(2)(3)(4)	(5) (7)	(6) (8)
0	0.0076	0.7896	0.8528
0.1	0.0082	0.2637	0.2844
0.5	0.0081	0.0724	0.0778
1	0.0080	0.0382	0.0407
1.5	0.0079	0.0261	0.0276
2	0.0079	0.0199	0.0209
3	0.0079	0.0136	0.0141
5	0.0079	0.0085	0.0085

**Case B: Normal Market**

$T$	(1)(2)(3)(4)	(5) (7)	(6) (8)
0	0.0079	0.7966	0.7984
0.1	0.0080	0.2655	0.2658
0.5	0.0080	0.0727	0.0724
1	0.0079	0.0382	0.0380
1.5	0.0080	0.0258	0.0256
2	0.0080	0.0195	0.0193
3	0.0080	0.0130	0.0129
5	0.0080	0.0078	0.0078

**Case C: Bear Market**

$T$	(1)(2)(3)(4)	(5) (7)	(6) (8)
0	0.0074	0.7960	0.7502
0.1	0.0076	0.2651	0.2504
0.5	0.0078	0.0718	0.0685
1	0.0079	0.0374	0.0358
1.5	0.0079	0.0251	0.0242
2	0.0079	0.0188	0.0182
3	0.0078	0.0125	0.0122
5	0.0079	0.0074	0.0074

Table 5.6: Implied Survival Probabilities of zero-recovery, Zero-Coupon Bonds for Several Maturities  $T = 0, 0.1, 0.5, 1, 1.5, 2, 3, 5$ . Three different market conditions: bull, normal and bear.

**Case A:Bull Market**

$T$	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
0	2.5000	2.0517	2.1077	2.1081	2.1632	2.1368	1.7877	
0.1	2.5000	2.0853	2.1090	2.1127	2.1628	2.1323	1.7848	
0.5	2.5000	2.1131	2.1041	2.1147	2.1645	2.0970	1.7904	
1	2.5000	2.1051	2.1034	2.1231	2.1547	2.0617	1.8010	
1.5	2.5000	2.0946	2.1033	2.1233	2.1500	2.0246	1.8033	
2	2.5000	2.1072	2.1027	2.1247	2.1495	1.9891	1.8023	
3	2.5000	2.1071	2.1017	2.1276	2.1483	1.9179	1.7926	
5	2.5000	2.1054	2.1038	2.1460	2.1330	1.7737	1.7907	

**Case B:Normal Market**

$T$	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
0	2.5000	2.4928	2.5490	2.5610	2.5551	2.5521	2.5552	
0.1	2.5000	2.5241	2.5479	2.5642	2.5548	2.5527	2.5546	
0.5	2.5000	2.5625	2.5506	2.5697	2.5482	2.5546	2.5576	
1	2.5000	2.5454	2.5520	2.5707	2.5575	2.5503	2.5605	
1.5	2.5000	2.5386	2.5498	2.5655	2.5551	2.5469	2.5615	
2	2.5000	2.5421	2.5509	2.5613	2.5552	2.5501	2.5619	
3	2.5000	2.5560	2.5513	2.5584	2.5567	2.5528	2.5601	
5	2.5000	2.5540	2.5590	2.5539	2.5660	2.5461	2.5507	

**Case C:Bear Market**

$T$	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
0	2.5000	2.8188	2.8777	2.8786	2.5983	2.5972	2.9261	
0.1	2.5000	2.8540	2.8775	2.8714	2.5985	2.6040	2.9262	
0.5	2.5000	2.9209	2.8833	2.8464	2.5965	2.6254	2.9289	
1	2.5000	2.9087	2.8776	2.8225	2.5989	2.6571	2.9202	
1.5	2.5000	2.9143	2.8793	2.7977	2.5963	2.6938	2.9285	
2	2.5000	2.9072	2.8787	2.7677	2.6052	2.7260	2.9325	
3	2.5000	2.9025	2.8844	2.7098	2.6000	2.7814	2.9359	
5	2.5000	2.8985	2.8816	2.5872	2.5904	2.9192	2.9408	

Table 5.7: Credit Spreads for Several Maturities and three different values of  $\epsilon$ : high  $\epsilon = 1/2$ , medium  $\epsilon = 1/4$  and low  $\epsilon = 1/16$ .

**Case A: Bull Market**

$T$	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
0	0.0093	0.9048	0.9135	0.9120	0.9143	0.9160	0.9324	
0.1	0.0087	0.3015	0.3048	0.3044	0.3050	0.3055	0.3110	
0.5	0.0087	0.0826	0.0829	0.0832	0.0830	0.0833	0.0848	
1	0.0087	0.0433	0.0433	0.0436	0.0433	0.0437	0.0444	
1.5	0.0088	0.0293	0.0293	0.0295	0.0294	0.0297	0.030	
2	0.0088	0.0222	0.0221	0.0222	0.0222	0.0224	0.0227	
3	0.0089	0.0149	0.0149	0.0149	0.0148	0.0151	0.0152	
5	0.0088	0.0091	0.0090	0.0090	0.0090	0.0092	0.0092	

**Case B: Normal Market**

$T$	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
0	0.0092	0.9053	0.8971	0.8978	0.8939	0.8937	0.8977	
0.1	0.0093	0.3019	0.2986	0.2992	0.2977	0.2979	0.2992	
0.5	0.0090	0.0821	0.0816	0.0816	0.0813	0.0815	0.0820	
1	0.0089	0.0429	0.0428	0.0427	0.0426	0.0428	0.0430	
1.5	0.0089	0.0291	0.0291	0.0290	0.0289	0.0290	0.0291	
2	0.0089	0.0221	0.0220	0.0220	0.0218	0.0219	0.0219	
3	0.0089	0.0148	0.0147	0.0148	0.0146	0.0147	0.0147	
5	0.0089	0.0088	0.0089	0.0089	0.0088	0.0089	0.0089	

**Case C: Bear Market**

$T$	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
0	0.0088	0.8908	0.8883	0.8659	0.8974	0.9054	0.8804	
0.1	0.0087	0.2969	0.2959	0.2886	0.2989	0.3020	0.2934	
0.5	0.0088	0.0807	0.0806	0.0791	0.0814	0.0820	0.0801	
1	0.0087	0.0421	0.0421	0.0416	0.0425	0.0430	0.0419	
1.5	0.0087	0.0285	0.0285	0.0283	0.0287	0.0290	0.0284	
2	0.0088	0.0215	0.0215	0.0214	0.0217	0.0218	0.0215	
3	0.0088	0.0145	0.0145	0.0145	0.0146	0.0146	0.0145	
5	0.0088	0.0087	0.0087	0.0089	0.0088	0.0088	0.0087	

Table 5.8: Prices of Zero-Coupon Bonds with Recovery for Several Maturities  $T = 0, 0.1, 0.5, 1, 1.5, 2, 3, 5$  and three different values of  $\epsilon$ : high  $\epsilon = 1/2$ , medium  $\epsilon = 1/4$  and low  $\epsilon = 1/16$ . Three different market conditions: bull, normal and bear.

**Case A: Bull Market**

$T$	(1)(2)(3)(4)	(5) (7)	(6) (8)
0	0.0085	0.7942	0.8324
0.1	0.0080	0.2650	0.2776
0.5	0.0079	0.0724	0.0757
1	0.0079	0.0381	0.0397
1.5	0.0079	0.0259	0.0268
2	0.0079	0.0197	0.0201
3	0.0079	0.0133	0.0135
5	0.0079	0.0082	0.0082

**Case B: Normal Market**

$T$	(1)(2)(3)(4)	(5) (7)	(6) (8)
0	0.0087	0.8004	0.8010
0.1	0.0081	0.2670	0.2670
0.5	0.0081	0.0725	0.0728
1	0.0081	0.0380	0.0380
1.5	0.0080	0.0256	0.0257
2	0.0080	0.0194	0.0194
3	0.0080	0.0130	0.0130
5	0.0079	0.0079	0.0078

**Case C: Bear Market**

$T$	(1)(2)(3)(4)	(5) (7)	(6) (8)
0	0.0072	0.7939	0.7889
0.1	0.0074	0.2650	0.2631
0.5	0.0076	0.0723	0.0719
1	0.0077	0.0380	0.0377
1.5	0.0078	0.0259	0.0254
2	0.0078	0.0196	0.0192
3	0.0078	0.0131	0.0129
5	0.0079	0.0080	0.0078

Table 5.9: Prices of zero-recovery, Zero-Coupon Bonds for Several Maturities  $T = 0, 0.1, 0.5, 1, 1.5, 2, 3, 5$  and three different values of  $\epsilon$ : high  $\epsilon = 1/2$ , medium  $\epsilon = 1/4$  and low  $\epsilon = 1/16$ . Three different market conditions: bull, normal and bear.

**Case A: Bull Market**

$T$	(1)(2)(3)(4)	(5) (7)	(6) (8)
0	0.0083	0.8033	0.8299
0.1	0.0077	0.2677	0.2762
0.5	0.0077	0.0732	0.0756
1	0.0078	0.0385	0.0396
1.5	0.0078	0.0261	0.0269
2	0.0078	0.0198	0.0204
3	0.0079	0.0135	0.0137
5	0.0079	0.0082	0.0082

**Case B: Normal Market**

$T$	(1)(2)(3)(4)	(5) (7)	(6) (8)
0	0.0083	0.7996	0.7926
0.1	0.0076	0.2667	0.2640
0.5	0.0078	0.0725	0.0722
1	0.0079	0.0379	0.0379
1.5	0.0079	0.0256	0.0256
2	0.0079	0.0194	0.0194
3	0.0079	0.0130	0.0130
5	0.0079	0.0078	0.0079

**Case C: Bear Market**

$T$	(1)(2)(3)(4)	(5) (7)	(6) (8)
0	0.0074	0.7955	0.7853
0.1	0.0077	0.2651	0.2619
0.5	0.0078	0.0722	0.0717
1	0.0078	0.0378	0.0374
1.5	0.0079	0.0256	0.0254
2	0.0079	0.0193	0.0192
3	0.0079	0.0129	0.0129
5	0.0079	0.0077	0.0079

Table 5.10: Implied Survival Probabilities of zero-recovery, Zero-Coupon Bonds for Several Maturities  $T = 0, 0.1, 0.5, 1, 1.5, 2, 3, 5$  and three different values of  $\epsilon$ : high  $\epsilon = 1/2$ , medium  $\epsilon = 1/4$  and low  $\epsilon = 1/16$ . Three different market conditions: bull, normal and bear.

**Case A: Bull Market**

$T$	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
5	2.5000	2.1453	2.1157	2.0211	1.8223	1.7195	1.4955	
6	2.5000	2.1362	2.1214	2.0058	1.8214	1.7048	1.4976	
7	2.5000	2.1376	2.1283	1.9894	1.8219	1.6884	1.4978	
8	2.5000	2.1326	2.1304	1.9746	1.8205	1.6726	1.4969	
10	2.5000	2.1289	2.1317	1.9440	1.8167	1.6430	1.4969	
12	2.5000	2.1448	2.1364	1.9142	1.8141	1.6111	1.4960	
15	2.5000	2.1433	2.1325	1.8656	1.8101	1.5711	1.4951	
20	2.5000	2.1359	2.1342	1.7905	1.8093	1.5056	1.4957	

**Case B: Normal Market**

$T$	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
5	2.5000	2.5465	2.5599	2.5736	2.5612	2.5528	2.5716	
6	2.5000	2.5458	2.5571	2.5729	2.5627	2.5526	2.5706	
7	2.5000	2.5430	2.5574	2.5711	2.5628	2.5549	2.5698	
8	2.5000	2.5488	2.5583	2.5684	2.5629	2.5559	2.5712	
10	2.5000	2.5533	2.5581	2.5702	2.5631	2.5569	2.5739	
12	2.5000	2.5605	2.5582	2.5659	2.5638	2.5583	2.5687	
15	2.5000	2.5571	2.5614	2.5618	2.5708	2.5545	2.5701	
20	2.5000	2.5603	2.5656	2.5610	2.5735	2.5518	2.5574	

**Case C: Bear Market**

$T$	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
5	2.5000	2.9032	2.8837	2.9948	3.3146	3.4172	3.7228	
6	2.5000	2.9014	2.8860	3.0164	3.3145	3.4335	3.7231	
7	2.5000	2.9123	2.8848	3.0365	3.3132	3.4570	3.7215	
8	2.5000	2.9075	2.8847	3.0573	3.3122	3.4736	3.7213	
10	2.5000	2.9110	2.8856	3.0980	3.3213	3.5195	3.7203	
12	2.5000	2.9089	2.8953	3.1392	3.3186	3.5604	3.7235	
15	2.5000	2.9021	2.8880	3.2018	3.3063	3.6244	3.7337	
20	2.5000	2.9012	2.8907	3.3087	3.3071	3.7340	3.7302	

Table 5.11: Credit Spreads for Several Higher Maturities  $T = 5, 6, 7, 8, 10, 12, 15, 20$ . (spreads in percentage points)



## Chapter 6

# Credit Risk Derivatives

### 6.1 Credit Risk Derivatives Overview

The advent of credit risk derivatives must be viewed against a foreground of a new paradigm of risk assessment and the application of financial derivatives to hedge these risks. Financial derivatives, in particular, allow the writer to create a market for specific features of an underlying independent of the underlying itself. For example, interest rate and currency derivatives can be used to hedge the risks of many of the features embedded in a bond. These traditional derivatives do not allow the trading of two important sources of risk, that is, the risk of default and the risk of fluctuations in marginal risk. These two features are just the more commonly encountered dimensions of credit risk. Credit risk derivatives are carefully modelled to isolate these types of risks and facilitate the trading in and hedging of these credit risks.

A financial derivative is termed a credit risk derivative if the value of the contingent claim of the contract is derived from the credit risk of the underlying financial instrument. Credit risk derivatives create a mechanism for investors to isolate credit risk from other forms of risks such as interest rate risk or market risk. The payoff of a credit risk derivative is triggered if a particular credit event occurs and usually the nature of the credit risk derivative selected originates from the underlying credit risk event.

In most instances credit risk derivatives are priced by inexplicably linking the credit quality of the underlying to the payoff function. This means, in general, credit risk derivatives are short-term in nature and can have a time to maturity of one to three years. As the credit risk derivatives market reaches maturity it is quite possible that instruments with longer time to maturity can be offered to investors.

The concept of credit risk derivatives was introduced in 1992 at a conference of the International Swap Dealers Association (ISDA) and since emerged as a useful risk management tool. There has been a phenomenal growth of credit risk derivatives worldwide. By the end of 1996 the international market size was estimated to be between \$100 billion to \$200 billion notional amount. The estimates of the British Bankers Association

(BBA) [117], for 1996, suggested that the size of the London market to be \$20 billion. If we compare the BBA estimate for the year 2000, to be \$900 billion, and their estimate for the year 2001, to be \$1.6 trillion, then clearly the worldwide market for credit risk derivatives has grown considerably. Notwithstanding the global market in credit risk derivatives has been forecast to rise to \$8.2 trillion by the end of 2006 according to a new report published by the BBA in September 2004. Nonetheless, the market for credit risk derivatives is still largely privately negotiated and remains off balance sheet and there is no tangible method to assess the size of the market without participants publishing that information.

The steady growth of the market of credit risk derivatives has potentially introduced new insights in the way credit risk is identified, priced and hedged. Credit risk derivatives are designed to diversify the credit risk exposure of portfolio's of bank loans or risky debt securities and allow credit markets to facilitate the transfer of credit risk from banks to those market participants best equipped to manage them. As the credit risk derivatives market has grown the products on offer have reached a level of maturity on their pricing methodology. This chapter explains how credit risk derivatives work and we set about explaining individual instruments (1) a total return swap (2) a credit default swap (3) a credit spread option. We subsequently price two credit risk derivatives illustrating different methodologies to determine a price.

The remainder of this chapter is organized as follows. First, we explain and illustrate three individual credit risk derivatives. In section 6.2 we describe the mechanism of a total return swap and give an example on how to hedge credit risk exposure via a total return swap. In section 6.3 we set out and explain the basic credit default swap structure and list opportunities to invest this credit risk instrument. We conclude our overview of credit risk derivative instruments in section 6.4 with an illustration of the basic structure and explanation of a credit spread option. We extend our overview of credit risk derivatives in section 6.5 to give an exposition of the Das and Sundaram (2000) [34] credit risk derivative model. We use this reduced form framework for the computation of a credit spread option. In section 6.6 we use the structural model framework to derive a closed form pricing formula for credit risky put option with one sided counterparty default risk based on the classic Merton (1974) [94] firm value model. We include closing remarks in section 6.7.

## 6.2 Total Return Swap

A total return swap is a widely used application of a credit risk derivative. To illustrate this concept we consider a market participant who wants to purchase a 3-year AA-rated bond issued by, say, UCT Corporation but does not want to bear the transaction costs and acquisition rights associated with the bond. Suppose, also, that a financial institution owns the same bond and is unable to extend credit to UCT Corporation since all its lines of credit is fully exhausted to UCT. Then in this example a total return swap will allow the market participant to receive the total cashflow return on this bond without actually

purchasing it. In addition, this will allow the financial institution to reduce its debt exposure to UCT Corporation as if it had sold the bond albeit no transaction has taken place.

### 6.2.1 Basic Structure of a Total Return Swap

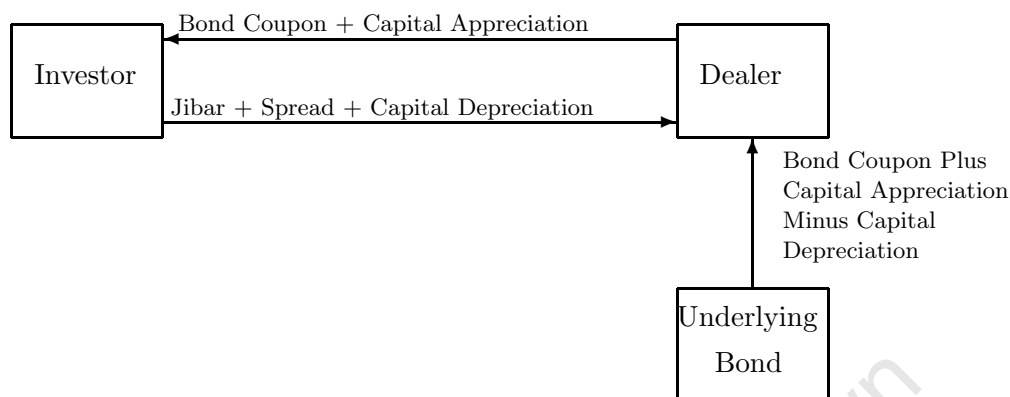


Figure 6.1: Basic Structure of a Total Return Swap

In Figure 6.1 we show an illustration of a total return swap. The swap contract agreed to by the counterparties defines the reference asset and its initial value  $P_0$ . The reference asset is usually a debt obligation such as a loan, a treasury bond or a corporate bond. The terms of the contract also defines the notional amount, the tenor of the swap and the reference rate. On the interest only leg of the swap the total return receiver commits to make a series of interest payments at a fixed reference rate. The interest payments  $I_1, I_2, \dots, I_T$  are made at periodic intervals with the first payment  $I_1$  determined at the commencement of the swap.

On the total return leg of the swap the interest payments on the bond is denoted by  $C$  and the total return receiver is entitled to this rate  $C$  of interest payments at the end of each period. In addition, at the end of the swap agreement the counterparties revalue the reference bond,  $P_T$ . If the value of the bond has increased to  $P_T > P_0$  then the total return receiver gains  $P_T - P_0$ . If the value of the bond has decreased to  $P_T < P_0$  then the total return receiver pays  $P_0 - P_T$ . Hence at maturity the total return swap has a final payment of  $C + P_T - P_0$ . Throughout the contract the dealer retains ownership of the bond but all the risk is transferred to the total return receiver.

Now, suppose that a credit event triggers the reference bond to go into default. Each counterparty to the agreement is released of its obligations to make the exchange interest payment that would be due at the end of the period in which default had occurred. However, the total return receiver is paid a final amount calculated on a specified recovery mechanism for the defaulted bond at the end of the default period.

### 6.2.2 Hedging Risk Exposure using a Total Return Swap

Many large financial institutions have loans or other debt securities on their balance sheet that they would like to hold to maturity for among other reasons relationship or regulatory purposes, but would like to hedge against the credit risk exposure that these assets bear. The credit risk exposure can be hedged if the financial institution enters into a total return swap in which it commits to pay a total return. As the counterparty responsible for the payment stream of the swap agreement the financial institution will determine its payments based on the credit quality of the borrower. If we consider our earlier example, in the context of Figure 6.1, where the borrower of the loan is also the obligor of UCT Corporation bonds and if UCT's credit rating weakens this would result in the final payment of  $P_T - P_0$  decreasing to the total return receiver. A total return swap agreement of this nature allows the financial institution to hedge its credit risk exposure while retaining the ownership of the loan and a decrease in  $P_T$  offsets the value of the loan. Further, the financial institution would receive a market rate of interest, say 3-month Jibar plus a spread while the counterparty would receive the total return payment stream coupled with the credit risk exposure of the loan.

## 6.3 Credit Default Swap

Credit default swaps are the vanilla instruments in the credit risk derivative market and its structure forms the basic building block for more complex instruments. The mechanism of a credit default swap is fairly straight forward. Suppose, a bank may wish to hedge itself against the credit risk exposure of a particular corporate and does so by selling risk exposure to a client. The bank pays a single up front premium in exchange for the clients obligation to make a payment on the occurrence of a defined credit default event. If at the end of the term the default swap agreement there is no default then the swap terminates with no additional financial obligation from either counterparty. In the event of default the client that has purchased the credit risk exposure has to fulfill his obligation to the bank. A credit default swap represents a form of credit insurance which is contingent on a specified credit event.

In Figure 6.2 we show an illustration of the basic structure of a credit default swap (CDS). In terms of a credit default swap agreement the counterparties agree on a notional amount, the reference entity, the tenor of the swap, the specified credit default event and the payment structure. In a CDS, the default protection buyer agrees to make a payment, or a series of payments, to the protection seller in exchange for a specified contingent payment should the reference entity experience a credit default event. If no credit event has occurred in the period spanning the tenor of the swap then the protection seller is released of his obligations.

### 6.3.1 Basic Structure of a Credit Default Swap

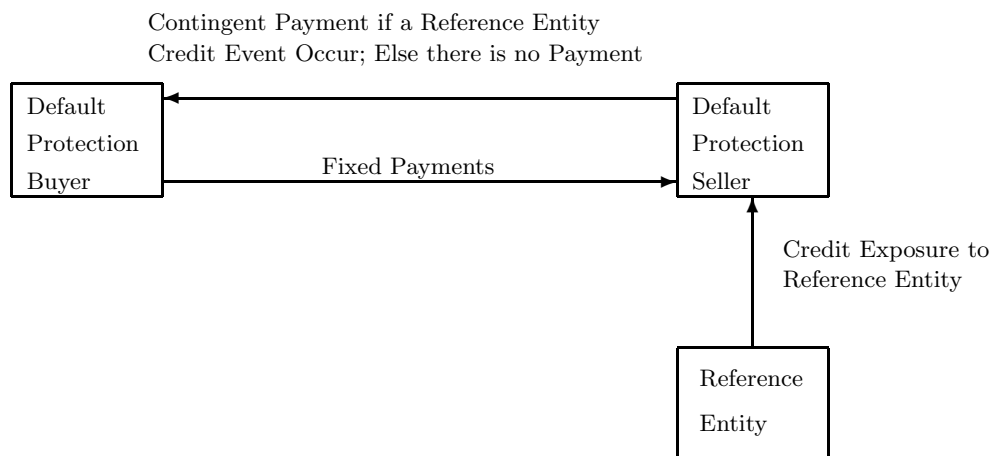


Figure 6.2: Basic Structure of a Credit Default Swap

### 6.3.2 Uses of Credit Default Swaps

Credit default swaps have many peculiar characteristics that offer attractive benefits to a range of participants in the credit markets. We give short summary of two interesting opportunities, investing credit and hedging in credit.

#### Investing in Credit

- CDS may be used to take a view on the weakening or strengthening in the credit rating of a reference entity.
- CDS can offer market participants an opportunity to invest in foreign markets without deriving any currency risk.
- Market participants can use CDS to design credit exposures to match their maturity requirements.

#### Hedging Credit

- CDS can be used confidentially to transfer the credit risk of loans without borrower consent and thus leaving client relationship intact.
- CDS can be used as effective short positioning instrument to buy credit protection for a specified tenor than to short the actual bonds.
- CDS are off-balance sheet instruments that can avoid the tax or accounting treatment that the sale of actual assets are subject to.

## 6.4 Credit Spread Option

An option confers on a buyer a right without an obligation to exercise that right. The credit spread is the differential yield on risky debt and government debt of the same tenor. A credit spread option is an option on a counterparty's credit spread. Treasury debt is normally considered the benchmark for default-free debt then any credit spread away from the benchmark is considered the premium, in yield, that market participants require to be compensated for the risk of default. The buyer of the option usually makes a one-off payment termed the option premium and the writer of the option in turn agrees to make a specified payment contingent on the event that the credit spread crosses a defined barrier.

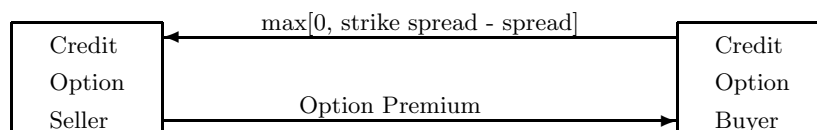


Figure 6.3: A Credit Spread Put Option

In Figure 6.3 we show an illustration of a credit spread put option. Suppose a market participant wants protection against the event that a particular bond's credit quality will weaken which in turn cause the credit spread to widen. A credit spread put option will provide the desired protection. Similarly, if a market participant considers a particular bond's credit quality will strengthen which in turn will cause the credit spread to tighten. A credit spread call can provide a higher return to investors without actually going long on the bond.

## 6.5 Das and Sundaram Credit Risk Derivative Model

In this section we present an overview of the credit spread option model as developed by Das and Sundaram (2000) [34] as an application of the reduced form modelling approach. In addition, we will use the methodology presented in the paper to compute a numerical value for a credit spread option using Microsoft Excel. The numerical results is stored in the hardware accompanying this dissertation.

The aim of the Das and Sundaram (2000) [34] paper is to present a model that can be implemented with ease and requires model inputs that are observable and available. The martingale pricing methodology is adopted and in particular a discrete-time reduced form model for valuing risky debt based on the term structure model of Heath, Jarrow and Morton (1990) [64] (hereafter discrete-time HJM) is developed. The discrete-time HJM methodology is uniquely adjusted to model risky debt by incorporating a *forward spread process* to the forward rate process for default riskfree bonds. Both the forward

spread process and the forward rate process are allowed to be correlated. Further, the probability of default at any time,  $t$ , is calibrated to the trajectory of the process up to time  $t$ . The model also assumes a recovery rate, in particular, the *recovery of market value* (RMV) mechanism as postulated by Duffie and Singleton (1999) [42]. If default occurs then the RMV condition applies which means that the zero coupon risky bond trades for a fraction  $x$  of its market value the instant before default.

The pricing lattice is essentially a path dependent no-arbitrage model that has an initial data set comprising (i) the term structures of risk-less forward rates and credit spreads and (ii) the associated term structures of volatilities of rates and spreads, respectively. They model the stochastic processes for risk-less forward rates and credit spreads by using the observable term structures of rates, spreads and the volatilities of these quantities and solve for the implicit risk-neutral drifts of the stochastic processes. This makes the discounted credit-risky security prices martingales. The RMV condition embeds economic consistency and it as well offers analytical tractability to the risk-neutral drifts by way of a recursive method. The computation of the model using the recursive equation system is consistent with the no-arbitrage assumption and is used to generate the lattice of forward rates and forward spreads. At each node the model computes default probabilities and recovery rates consistent with the credit spreads. The default probabilities are indirectly modelled using a logit equation.

The arbitrage-free credit risk derivative model has several attractive features. The model takes as an input observed credit spreads as such there is no need to derive implied credit spreads from default probabilities and recovery rates and to calibrate the implied spreads to the observed spreads. This makes for ease of computation of credit risk derivatives whose payoffs are contingent on the spread. The discrete-time HJM framework is the underlying methodology of the model it makes use of forward rates and *forward spreads* of risk-less and risky debt. Together with the RMV condition the model leads to a recursive structure of the risk-neutral drifts of the forward rate and forward spread processes. In addition, the recursive structure facilitates the computational tractability of the path dependence feature of the model. The recursion also generates cumulative default probabilities at each node which are needed to price a credit default swap.

The remainder of the model overview is organized as follows. Section 6.5.1 states the underlying assumptions and describes the model. Section 6.5.2 states the main results of the derivation of the recursion equations for the risk-neutral drifts. Section 6.5.3 describes the recursive representation of the risky bond prices while Section 6.5.4 describes the logit equation approach to modelling default probabilities. Section 6.5.5 describes the actual implementation of the model and illustrates by pricing a credit risk derivative. Section 6.5.6 completes the overview with a few concluding comments.

### 6.5.1 The Model Setup

The model is developed in a discrete-time HJM framework and options with path dependent features will be priced using computer based tools. The model economy is on

a finite time interval  $[0, T^*]$  with periods of length  $h > 0$ . The economy has arbitrage free markets and allows for all maturities of both risk-less zero coupon bonds and risky zero coupon bonds. An equivalent martingale measure  $\mathbb{Q}$  characterizes this economy and all calculations that follow will be taken with respect to this measure. The forward rate is defined as  $f(t, T)$  on an arbitrary time interval  $(t, T)$ , with  $0 \leq t \leq T^* - h$ , is the rate determined at time  $t$  for risk-less bonds over the interval  $(T, T + h)$ . In the discrete-time economy the forward rate for risk-less bonds is assumed to evolve as follows:

$$f(t + h, T) = f(t, T) + \alpha(t, T)h + \sigma(t, T)X_1\sqrt{h} \quad (6.1)$$

with  $\alpha$  the drift and  $\sigma$  the volatility coefficient of the process, respectively.  $X_1$  is a random variable with equal probability of taking on values  $\pm 1$ . Similarly, we denote forward rates for credit-risky bonds as  $\varphi(t, T)$  and hence we can define the forward spread on risky bonds as:

$$s(t, T) = \varphi(t, T) - f(t, T)$$

The evolution of the forward spreads is defined as:

$$s(t + h, T) = s(t, T) + \beta(t, T)h + \eta(t, T)X_2\sqrt{h} \quad (6.2)$$

with  $\beta$  the drift and  $\eta$  the volatility coefficient of the process, respectively.  $X_2$  is a random variable with equal probability of taking on values  $\pm 1$ . Both  $X_1$  and  $X_2$  are arbitrarily correlated random variables that can be assigned values  $\pm 1$  with model consistent probabilities. The risk-less zero coupon bond of maturity  $t \leq T$  is defined by the pricing equation:

$$P(t, T) = \exp \left\{ - \sum_{k=\frac{t}{h}}^{\frac{T}{h}-1} f(t, kh) h \right\} \quad (6.3)$$

Similarly, we can define the risky zero coupon bond is defined as:

$$\Pi(t, T) = \exp \left\{ - \sum_{k=\frac{t}{h}}^{\frac{T}{h}-1} \varphi(t, kh) h \right\} \quad (6.4)$$

The spreads on the risky bonds is a measure of the cost of default and imputed in this value is both the probability of default and the value of the bond upon default. The probability of default in one time interval  $t + h$  is denoted as  $\lambda(t)$ .

The theme of this paper is to accomplish the pricing of risky debt in a risk-neutral framework. This is computed in the steps that follow. First, the risk-less interest rates lattice is generated by solving for the risk-neutral drifts,  $\alpha$ . This ensures a no-arbitrage interest rate lattice. Second, the forward spread process is generated by solving for the risk-neutral drifts,  $\beta$  and the credit spread lattice is superimposed on the first lattice. Finally, together with the recursive feature and the default rate process the implementation of the model is illustrated. Next, the risk-neutral drifts are identified.



### 6.5.2 Identifying the Risk Neutral Drifts

In this section we state the main results of the derivation of the recursive expressions for the drifts  $\alpha$  and  $\beta$  of the forward rate and forward spread processes, respectively, in terms of the volatilities  $\sigma$  and  $\eta$ . The first recursive expression relating the risk-neutral drifts,  $\alpha$ , to the volatilities,  $\sigma$ , at each time  $t$  is stated as:

$$\sum_{k=\frac{t}{h}+1}^{\frac{T}{h}-1} \alpha(t, kh) = \frac{1}{h^2} \ln \left( \mathbb{E}_t \left[ \exp \left\{ - \sum_{k=\frac{t}{h}+1}^{\frac{T}{h}-1} \sigma(t, kh) X_1 h^{3/2} \right\} \right] \right) \quad (6.5)$$

The second recursive relation defines  $\alpha$  and  $\beta$  in terms of  $\sigma$  and  $\eta$  is stated as:

$$\begin{aligned} & \exp \left\{ \sum_{k=\frac{t}{h}+1}^{\frac{T}{h}-1} [\alpha(t, kh) + \beta(t, kh)] h^2 \right\} \\ &= \mathbb{E}_t \left[ \exp \left\{ - h^{3/2} \sum_{k=\frac{t}{h}+1}^{\frac{T}{h}-1} [\sigma(t, kh) X_1 + \eta(t, kh) X_2] \right\} \right] \end{aligned} \quad (6.6)$$

The recursive equations derived in terms of the risk-neutral drifts  $\alpha$  and  $\beta$  facilitates the analytical tractability of the model.

### 6.5.3 A Recursive Representation of Risky Bond Prices

Similar, to the risk-neutral drifts the term structure of risky bond prices can also be stated in terms of a recursive relationship. The recursive representation is in terms of bond prices of short maturities and is given as follows:

$$\begin{aligned} \Pi(t, T) &= \Pi(t, t+T) \cdot \mathbb{E}^t[\Pi(t+h, T)] \\ &= \Pi(t, t+T) \cdot \mathbb{E}^t[\Pi(t+h, t+2h)] \\ &\quad \cdot \mathbb{E}^{t+h}[\Pi(t+2h, t+3h)] \\ &\quad \cdot \mathbb{E}^{t+2h}[\dots] \end{aligned} \quad (6.7)$$

Clearly, the recursive representation of prices of risky bonds plays a central role in the analytical tractability of the model.

### 6.5.4 Towards Implementation of the Model

In this section, the quantities that have remained unspecified are made more precise. They were chosen for their simplicity both in exposition and implementation. In particular, the quantities described here are the random variables  $X_1$  and  $X_2$  and the default probability  $\lambda(t)$ . The model makes the canonical discrete-time assumption that  $X_1$  and  $X_2$  are binomial random variables and are allowed to be assigned values of  $\pm 1$

with equal probability. In addition,  $X_1$  and  $X_2$  are assumed to be arbitrarily correlated with coefficient  $\rho$  with joint distribution  $(X_1, X_2)$  given as:

$$(X_1, X_2) = \begin{cases} (+1, +1) & \text{w.p. } \frac{(1+\rho)}{4} \\ (+1, -1) & \text{w.p. } \frac{(1-\rho)}{4} \\ (-1, +1) & \text{w.p. } \frac{(1-\rho)}{4} \\ (-1, -1) & \text{w.p. } \frac{(1+\rho)}{4} \end{cases}$$

Empirical regularities suggest that spreads and interest rates are usually positively correlated, however, under different economic regimes this assertion may not be so telling. The following equation is a fundamental relationship that links the short spread  $s(t, t)$  to the default probability  $\lambda(t)$  and the recovery rate  $\phi(t)$ .

$$s(t, t) = -\frac{1}{h} \ln [1 - \lambda(t) + \lambda(t)\phi(t)] \quad (6.8)$$

An additional equation is required to decompose the short spread into its constituent components. The restriction on the second equation is that  $\lambda$  is a probability and as such it has to reside in the interval  $[0, 1]$ . The logit equation is a suitable representation for the probability function and is given as :

$$\lambda(F, S) = \frac{1}{e^{-x} + 1} \quad , \quad x = a + b \cdot F + c \cdot S \quad (6.9)$$

with  $F$  and  $S$  representing the term structures of the forward and spread curves, respectively.

It is instructive to note estimates of the parameters of (6.9) are set in the real world which in-turn implies a real world probability of default  $\lambda^P(t)$ . A translation from the real world to the risk-neutral world is proffered and the risk neutral probability of default  $\lambda$  can be expressed in terms of  $\lambda^P$  and the risk premium  $\xi$  as:

$$\lambda(t) = \lambda^P(t) \left[ \frac{1 - \exp \{ -s(t, t)h \}}{1 - \exp \{ -(s(t, t) - \xi(t))h \}} \right] \quad (6.10)$$

Clearly, for  $\xi > 0$  we confirm the intuitive condition  $\lambda > \lambda^P$ . These expressions were used to extract the parameters of (6.9). Table 6.1 quotes the parameters as presented in the paper.

### 6.5.5 Implementation of the Model

In this section we describe the discrete-time engineering implementation of the model using a non-recombining lattice. Along the interest rate and spread process we obtain a double-binomial structure with four branches emanating from each node of the lattice. The risk-neutral drifts  $\alpha$  and  $\beta$  can be computed at any time  $t$  and hence the values

<i>Parameter</i>	<i>Estimate</i>
a	5.44
b	-10.43
c	-27.24
corr	0.86
$R^2$	0.58

Table 6.1: Parameters of the Default Probability

for the forward curves for interest rates and spreads can be readily obtained. Further, given the curves  $f(t, T)$  and  $s(t, T)$ , respectively, we may compute the one-period default probability  $\lambda(t)$  and the recovery rate  $\phi(t)$  at each node. In summary, we have information related to all three risks that is necessary to evaluate risky debt, interest rates, default probabilities and recovery rates. The sample lattice with this information appears in Figure 6.4 where  $p$  and  $q$  represents  $\frac{(1+\rho)}{4}$  and  $\frac{(1-\rho)}{4}$ , respectively.

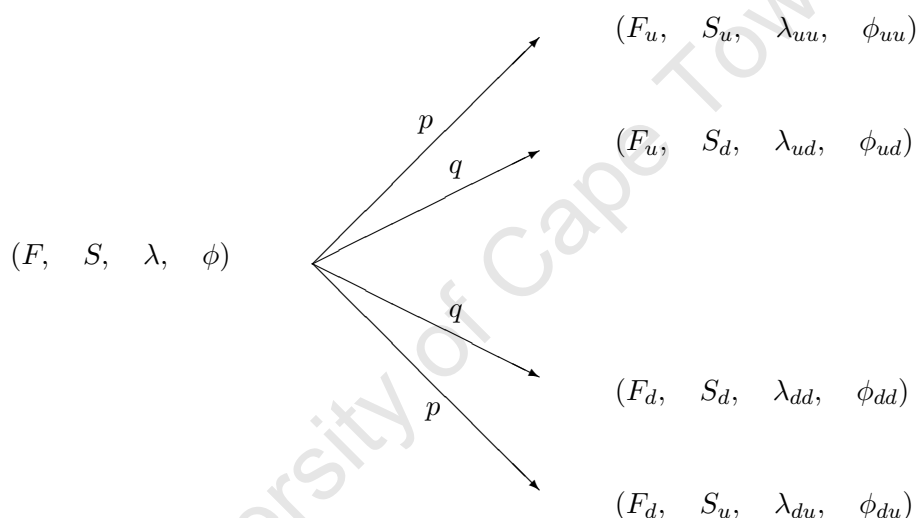


Figure 6.4: Information generated at each node in the branching process

A recursive algorithm is a popular and efficient computational technique to generate the lattice as shown in Figure 6.4. Option pricing regularly uses this technique and is particularly useful in the scheme described above since the recombination of branches at each node is not a restriction.

At each node we have no-arbitrage conditions and hence each sub-tree on the lattice maybe treated singularly. If the values at any arbitrary node is known then this allows the recursion procedure to generate the next period values. We will illustrate this method by pricing a credit spread option.

### 6.5.6 Numerical Example: Credit Spread Option

Thus far in this dissertation we have shown models of credit risk where a counterparty to a contract can probably default on its obligations. For example, the obligor of a corporate bond may default on a interest payment or the principal payment. The research presented in this section shows a different type of credit risk model where the payoff on a contingent claim is determined by, more specifically, the degrading in the credit quality of the underlying security in contrast to a default based model. An example of this type of contingent claim is a credit spread option where the payoff on this instrument is contingent on the credit spread widening.

We aim to price a credit spread call option as an application of the Das and Sundaram (2000) [34] credit risk derivative model. Formally, a credit spread call option on a, say, defaultable bond  $\Pi(t, T)$  with maturity  $t^* < T$  and strike spread  $K$  gives the holder the right to buy the credit risky bond at time  $t^*$  at a price that corresponds to a yield spread of  $K$  above the yield of an otherwise identical risk-free bond  $P(t, T)$ . The payoff to this contract is:

$$100 * \max[0, s(t^*, T) - K]$$

where the payoff has a par value of 100 and  $s(t^*, T)$  is the spread differential between the risky bond and risk-less bond at time  $t^*$ . Credit spread call options allow protection against fluctuations in risk-free interest rates and the associated movements in credit spreads. For example, you can hedge a short position in mark-to-market exposure of changes in spreads with credit spread call options.

Now, let us price a European credit spread call option with maturity  $t^* < T$  and exercise price  $K$  on a credit risky zero coupon bond with maturity  $T$  and denote its value by  $CSO$ . Moreover, we will implement the example of a credit spread option shown in the Das and Sundaram (2000) [34] paper. We use Microsoft Excel to perform the computation and the input data is shown in Table 6.2 as follows:

Period	$T$	$(T - h, T)$	$f(0, T)$	$\sigma_f$	$s(0, T)$	$\sigma_s$
1	0.5	(0, 0.5)	0.06	0.015	0.010	0.005
2	1.0	(0.5, 1)	0.07	0.012	0.015	0.006
3	1.5	(1, 1.5)	0.08	0.011	0.020	0.007
4	2.0	(1.5, 2)	0.09	0.010	0.022	0.008

Table 6.2: Input Data for the Credit Spread Option

The option strike spread is 0.015 and has a time to expiration of 3 periods with each period half-year in length. The contract notional value is taken to be 100 units of currency.

The lattice was constructed using the methodology described in the previous sections. The valuation of the  $CSO$  along the lattice in this model is fairly straightforward. At

each node of the lattice, the forward rates and the forward spreads are available in the information set at the node. This determines the forward rates and forward spreads scheduled for the next time period. This process is recursively implemented until the information set at each terminal node is computed. Thus applying the boundary condition of the  $CSO$  at each terminal node and discounting the spread prices in the default and non-default states, and moving backwards along the lattice yields the price of the  $CSO$ . The price of the option amounts to  $CSO = 0.095$ . The pricing lattice for the credit spread option is shown in Table 6.3.

### 6.5.7 Supplementary Remarks

This paper sets out the discrete-time no-arbitrage framework that takes as inputs observables to price credit risk derivatives. The model has some attractive features in that it handles path dependence and it can be adapted to price a range of credit instruments. The computational implementation of the model exploits the recursive technique which is useful to compute complex derivatives. The model utilises simple concepts and observable inputs that closes the no-arbitrage condition and as such offers scope to alternative concepts to improve the model.

$t = 1$	$t = 2$	$t = 3$	$t = 4$
0.000949895	1	1	$uu$ 0.007755769 $ud$ 0 $du$ 0.007755769 $dd$ 0  $uu$ 0 $ud$ 0 $du$ 0 $dd$ 0
			$uu$ 0.00372583 $ud$ 0  $du$ 0.00372583 $dd$ 0  $uu$ 0.007755769 $ud$ 0 $du$ 0.007755769 $dd$ 0  $uu$ 0 $ud$ 0 $du$ 0 $dd$ 0
			$uu$ 0.000684701 $ud$ 0 $du$ 0.000684701 $dd$ 0  $uu$ 0 $ud$ 0 $du$ 0 $dd$ 0
			$uu$ 0 $ud$ 0 $du$ 0 $dd$ 0
			$uu$ 0.000328927 $ud$ 0  $du$ 0.000328927 $dd$ 0  $uu$ 0.000684701 $ud$ 0 $du$ 0.000684701 $dd$ 0  $uu$ 0 $ud$ 0 $du$ 0 $dd$ 0
			$uu$ 0.001798841 $ud$ 0.000158807  $du$ 0.001798841 $dd$ 0.000158807
			$uu$ 0.007755769 $ud$ 0 $du$ 0.007755769 $dd$ 0  $uu$ 0 $ud$ 0 $du$ 0 $dd$ 0
			$uu$ 0.00372583 $ud$ 0  $du$ 0.00372583 $dd$ 0  $uu$ 0.007755769 $ud$ 0 $du$ 0.007755769 $dd$ 0  $uu$ 0 $ud$ 0 $du$ 0 $dd$ 0
			$uu$ 0.000684701 $ud$ 0 $du$ 0.000684701 $dd$ 0  $uu$ 0 $ud$ 0 $du$ 0 $dd$ 0
			$uu$ 0 $ud$ 0 $du$ 0 $dd$ 0
			$uu$ 0.000328927 $ud$ 0  $du$ 0.000328927 $dd$ 0  $uu$ 0.000684701 $ud$ 0 $du$ 0.000684701 $dd$ 0  $uu$ 0 $ud$ 0 $du$ 0 $dd$ 0
			$uu$ 0.001798841 $ud$ 0.000158807  $du$ 0.001798841 $dd$ 0.000158807
			$uu$ 0.007755769 $ud$ 0 $du$ 0.007755769 $dd$ 0  $uu$ 0 $ud$ 0 $du$ 0 $dd$ 0
			$uu$ 0.00372583 $ud$ 0  $du$ 0.00372583 $dd$ 0  $uu$ 0.007755769 $ud$ 0 $du$ 0.007755769 $dd$ 0  $uu$ 0 $ud$ 0 $du$ 0 $dd$ 0
			$uu$ 0.000684701 $ud$ 0 $du$ 0.000684701 $dd$ 0  $uu$ 0 $ud$ 0 $du$ 0 $dd$ 0
			$uu$ 0 $ud$ 0 $du$ 0 $dd$ 0
			$uu$ 0.000328927 $ud$ 0  $du$ 0.000328927 $dd$ 0  $uu$ 0.000684701 $ud$ 0 $du$ 0.000684701 $dd$ 0  $uu$ 0 $ud$ 0 $du$ 0 $dd$ 0
			$uu$ 0.001798841 $ud$ 0.000158807  $du$ 0.001798841 $dd$ 0.000158807
			$uu$ 0.007755769 $ud$ 0 $du$ 0.007755769 $dd$ 0  $uu$ 0 $ud$ 0 $du$ 0 $dd$ 0

Table 6.3: Pricing Lattice for a Credit Spread Option

## 6.6 A Firm Value Model for Credit Risk Derivatives

In the credit risk models we present in this dissertation we make the implicit assumption that the writers of options and derivatives are credit worthy and bear no risk of default. While this assumption is plausible if we consider options traded in a regulated exchange where participants are guaranteed potential payoffs on their option positions it may not always be valid for options traded in the over-the-counter market. In standard business transactions between counterparties the risk of default or non-performance by either counterparty is of major concern and especially impacts on the pricing of financial instruments and over-the-counter contracts.

These foreground observations suffice to make explicit the assumption of *one-sided counterparty default risk*, that is, we need to consider the valuation of derivatives where the writers of these financial instruments can potentially default on their contingent liabilities. For example, consider an option written on a futures contract with no risk of default associated with the futures contract. The writer of the option is susceptible to default risk since he may not be able to honor the contingent payoff of the exercised option.

Intuitively, credit risky options are deemed to be priced cheaper than their identical risk-less counterparts. This has an important empirical implication for both counterparties in the over-the-counter derivatives transactions. More precisely, current accounting practice stipulates that corporates report their positions in financial instruments at fair value on their balance sheet. As a result investors can improve their analysis of the risk structure of the corporate. On the other hand, this also assists corporates, banks and other financial institutions to improve their risk management practices to meet the capital adequacy requirements of the new Basel Capital Accord (2004) [11].

The aim of this section is to propose a traditional Merton (1974) [94] firm value model as the underlying framework to derive an analytic pricing formula for the value of a credit risky European put option where the nominal value of the claim recovered at default is specified as a fraction of the counterparty firm value relative to the single issue of debt. In addition, we specify that the option is written on equity as the underlying asset and the credit risk originates from the correlation of the counterparty assets of the firm and equity. This model contributes to literature by demonstrating the ability to capture counterparty risk and improve on the valuation of Black-Scholes (1973) [20] type credit risky contingent claims.

This section is organized as follows. Section 6.6.1 gives a concise overview of a sample of literature relevant to our modelling framework. The model setup is presented in section 6.6.2. In section 6.6.3 we develop a firm value model for a credit risky contingent claim. Section 6.6.4 shows a closed form pricing formula for a credit risky put option. We analyze the credit risky put option by showing numerical examples in section 6.6.5. We close with supplementary remarks in 6.6.6.

### 6.6.1 Overview of Literature for the Firm Value Model

The two main categories for pricing credit risk are the structural model and the reduced-form model methodologies. In some cases these modelling approaches form the basis for the pricing of credit risk derivatives. While we price a credit risky put option in the structural model framework a survey of literature shows that both modelling approaches are adopted to price derivatives with counterparty risk. We will give a concise overview of the research by Johnson and Stulz (1987) [76], Hull and White (1995) [71] and Klein (1996) [84] as to underscore the origins of our research.

In structural models the capital structure generally comprise equity and debt with the value of equity usually representing a proxy for the assets of the firm. In chapter 3 we demonstrate for the pricing of credit risk that default occurs when the value of the assets of the firm breaches a lower default threshold where the threshold is a function of a zero coupon credit risky bond. Similarly, in their research Johnson and Stulz (1987) [76] price credit risky options in the structural model framework albeit with the modification that debt is represented by the value of a single option and default occurs when the counterparty assets is of lower value than the payoff of the option at maturity. At default the holder of the option receives all the counterparty assets. In choosing the liability of the firm as the value of the option they implicitly assert that the default threshold is stochastic in nature. The debt of the firm represented by a single option is a reasonable assumption only if the potential payoff of the option is a small proportion of the total value of the firm. On the other hand, this assumption can lead to model mis-specification if the option value represents a sizeable portion of the option writer's assets at the start of the option's tenure. As a consequence, credit risk may arise from both a decrease in the counterparty assets and a considerable increase in the value of the option. Nonetheless, the appealing features of this model is that at default the recovery value of the option is a function of the counterparty asset value. In addition, the value of the asset underlying the option is correlated with the value of the assets of the counterparty. Like in the Merton (1974) [94] model Johnson and Stulz (1987) [76] consider a single liability in their capital structure which is not generally representative of standard capital structures and in turn limits the applicability of their model.

Hull and White (1995) [71] propose a model for both American and European credit risky securities with the assumption that default of the option writer can occur both at maturity of the option and as in a first-passage time setting. Unlike the Johnson and Stulz (1987) [76] model where the option is a single claim on the counterparty assets Hull and White (1995) [71] relax this assumption to include other claims to rank equally with the option at bankruptcy. They generalize the default boundary such that the expected loss at default is a random proportion of the no-default value of the option while the pricing of the credit risky security is consistent relative to its risk-less counterpart. As a base case example they show an analytical pricing formula for a long position in a European option which is adapted to value other credit risky securities. In addition, they implement the model numerically for foreign currency European and American call options subject



to default risk. American options are less affected by the impact of default risk than European options since American options can be exercised when the value of the assets of the firm is above the default threshold. It is instructive to note that Hull and White (1995) [71] consider in their model the assumption that the asset underlying the option is independent of the assets of the option writer. They invariably concede that this could be a restrictive assumption and in particular may be applicable to institutions with large assets relative to a contingent option payoff.

The research by Klein (1996) [84] extends the Johnson and Stulz (1987) [76] approach by modelling credit risk in the Merton (1974) [94] framework with the additional assumption of multiple liabilities in the counterparty capital structure. In the event of default the expected loss is a proportion of the nominal claim on the assets of the counterparty but explicitly relates the recovery value to the firm value of the counterparty. In addition, the model proposes that the assets of the counterparty and the asset underlying the option are correlated. While the recovery at default claim is paid out at option maturity default is allowed to occur during the option tenure with the model encompassing the realistic feature of calculating the value of assets for distribution to creditors at the option maturity date. Klein (1996) [84] calculates the values for credit risky call options for varying parameter values and compares them to Black-Scholes (1973) [20] and Johnson and Stulz (1987) [76] values. Since the Johnson and Stulz (1987) [76] and Klein (1996) [84] model assumptions considerably differ a direct comparison of option values cannot be performed. Nonetheless, it is observed that the option values in Klein (1996) [84] are generally less than that calculated in Johnson and Stulz (1987) [76]. Klein (1996) [84] suggests that this could be attributed to the restricted model emphasis for compensating for the increase in option value.

### 6.6.2 The Model Setup

Consider an economy over the time interval  $[0, T]$  where the distribution of events is modelled by the probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . The information flow to the economy includes the default information and is modelled by the filtration  $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ . The firm value process is specified to be ex-dividend. We will refer to  $T$  as the maturity date. In addition, we suppose that the economy is complete such that in terms of the risk-neutral valuation methodology there exists a unique probability measure  $\mathbb{Q}$  equivalent to the objective probability measure  $\mathbb{P}$  and all modelling is done under this risk-neutral martingale measure.

The basic assumptions of this model follow those from the Black-Scholes (1973) [20] framework and as such allows for consistency in the application of option pricing techniques to the Merton (1974) [94] firm value model. In addition, this setup constructs the context to value credit risky European options.

**Assumption 1:** Capital markets are frictionless with the absence of administrative costs and taxes. We have continuous time trading of assets and these assets are deemed per-

fectly divisible. Corporate insiders and investors have perfect information and this makes borrowing rates equal to lending rates.

**Assumption 2:** There are sufficiently many buyers and sellers of assets that are willing to trade assets at the market price. In addition, there are no arbitrage opportunities in these trades.

**Assumption 3:** The firm's capital structure is defined by two types of claims, risky debt,  $D$ , and equity,  $S$ . The value of the firm is given as:

$$V_t = D_t + S_t$$

In addition, we assume that the value of the firm is equal to the value of the assets of the firm.

**Assumption 4:** At the start of the firm value process we have a fixed option contract and the firm is in a non-default state.

**Assumption 5:** A risk-less asset is defined with a constant rate of interest per unit of time. For example, we have a risk-free bond that pays 1 unit of currency at maturity,  $T$ , is defined as  $P(0, T) = \exp(-rT)$ , where  $r$  is the risk-less rate of interest.

**Assumption 6:** The value process  $V$  of the option writer's (counterparty) assets of the firm follow a geometric Brownian motion and is given as:

$$dV_t = \mu_V V_t dt + \sigma_V V_t dB_V(t)$$

with  $\mu_V$  being the instantaneous expected rate of return on the assets of the firm,  $\sigma_V$  is the instantaneous standard deviation of the return on the assets of the firm and  $B_V(t)$  a standard Brownian motion. All parameters are defined under the objective probability measure,  $\mathbb{P}$ . In addition,  $\sigma_V$  is assumed to have the additional characteristic of being constant over time.

**Assumption 7:** The share price  $S$  is the underlying asset of the option and its dynamics follow a geometric Brownian motion and is given as:

$$dS_t = \mu_S S_t dt + \sigma_S S_t dB_S(t)$$

with  $\mu_S$  being the instantaneous expected rate of return on the share price,  $\sigma_S$  is the instantaneous standard deviation of the return on the share price and  $B_S(t)$  a standard Brownian motion. All parameters are defined under the objective probability measure,  $\mathbb{P}$ . In addition,  $\sigma_S$  is assumed to have the additional characteristic of being constant over time. Moreover, the constant instantaneous correlation coefficient  $\nu$  describes the interaction between Brownian motions driving the asset and share price processes.

**Assumption 8:** Default occurs at maturity,  $T$ , of the option only if the value  $V_T$  of the counterparty assets is less than the threshold value  $\bar{P}_T$ , where  $\bar{P}_T = \max(X - S_T, 0)$ .  $X$  represents the strike price of the option and  $S_T$  represents the share price underlying the option at maturity.

**Assumption 9:** There are no costs associated with bankruptcy nor are there any cash outflows during the period of the debt contract.

**Assumption 10:** We strictly adhere to the absolute priority rule and equity owners have a claim on the assets of the firm only if bondholders have been fully reimbursed.

**Assumption 11:** At default the option holder receives  $\omega$  times the default-free claim where  $\omega$  represents the recovery rate.

**Assumption 12:** *One-sided default risk.* There is only default risk associated with the writer of the option.

These assumptions are consistent with the research by Johnson and Stulz (1987) [76] and Ammann (2001) [4] where default can occur only at maturity of the option. Additionally, we allow for debt to be a single option or contingent liability written on the share price of the counterparty. In practice, it is generally accepted that a negative event, for example, a credit rating downgrade can adversely affect the share price of a public corporate. Further, we note that the market value of a corporate is inexplicably linked to the price of its traded shares to the extent we make the reasonable assumption that the assets of the counterparty is correlated with its share price. This setup has the advantage of relating the collateral of the firm to the recovery rate of the contingent claim in default. These assumptions are appropriate since it captures certain relevant features of empirical occurrences and aims to improve the valuation of Black-Scholes (1973) [20] type credit risky options.

### 6.6.3 A Firm Value Model for a Credit Risky Contingent Claim

The firm has a capital structure that consists of two components, risky debt and equity. The corporate debt is assumed to be a single European option  $\bar{P}$  due at maturity  $T$ . We formally adopt the following definitions:

- $V_t$  = value of the firm's assets at time  $t \in [0, T]$
- $D_t$  = value of the firm's debt at time  $t \in [0, T]$
- $S_t$  = value of the firm's equity at time  $t \in [0, T]$

We suppose that the firm is a public entity with an observable share price process such that we model this process as a suitable proxy for the firm value process. In addition, at every time  $t \in [0, T]$  we observe that

$$V_t = S_t + D_t$$

Pricing under the risk-neutral probability measure  $\mathbb{Q}$  we define the dynamics of the firm value process and the share price process to evolve according to a geometric Brownian motion.  $V_t$  and  $S_t$  can be expressed as:

$$dV = rVdt + \sigma_V V d\tilde{B}_V(t) \quad (6.11)$$

$$dS = rSdt + \sigma_S S d\tilde{B}_S(t) \quad (6.12)$$

where  $r$ ,  $\sigma_V$  and  $\sigma_S$  are suitably chosen such that the expressions in (6.11) and (6.12) are well defined.  $\tilde{B}_V(t)$  and  $\tilde{B}_S(t)$  are assumed to be correlated Brownian motions with  $\nu$  being defined as the correlation coefficient.  $S$  and  $V$  follow a bivariate normal distribution. The instantaneous expected return of each process is exactly the risk-less rate of return  $r$ . Further, under risk-neutral dynamics we define the risk-less rate of interest,  $r$ , and the instantaneous standard deviation of the returns,  $\sigma_V$  and  $\sigma_S$ , to be constant.

We define  $\bar{P}(T, T)$  to be an arbitrary terminal payoff by a counterparty firm at the end of the time horizon  $T$ . Moreover, it is assumed that the contingent claim has European features and can only be exercised at maturity  $T$ . As shown in Johnson and Stulz (1987) [76] we can construct a perfect hedge for a credit risky claim. For the two sources of randomness  $\tilde{B}_V(t)$  and  $\tilde{B}_S(t)$  we can construct a portfolio of one long position in  $\bar{P}$  and two short positions in some quantities of the underlying assets  $S$  and  $V$ . The price of the credit risky contingent claim must satisfy the following two factor PDE:

$$\frac{\partial \bar{P}}{\partial t} + \frac{1}{2} \sigma_S^2 S^2 \frac{\partial^2 \bar{P}}{\partial S^2} + \frac{1}{2} \sigma_S \sigma_V \nu S V \frac{\partial^2 \bar{P}}{\partial S \partial V} + \frac{1}{2} \sigma_V^2 V^2 \frac{\partial^2 \bar{P}}{\partial V^2} + rS \frac{\partial \bar{P}}{\partial S} + rV \frac{\partial \bar{P}}{\partial V} - r\bar{P} = 0 \quad (6.13)$$

In general the value of any contingent claim dependent on  $S$  and  $V$  can be obtained by solving the PDE given by expression (6.13) subject to specified boundary conditions.

The nature of the derivative we aim to price is based on the solvency state of the counterparty firm. The solvency level of the firm at maturity must be such that  $V_T \geq D_T$  for no default to occur else the firm is said to be in default if  $V_T < D_T$ . If the firm remains solvent during the tenure of the claim then the payoff of the contingent claim is equal to the payoff of a default-free contingent claim  $P(T, T)$ . Alternately, if the firm goes into an insolvent state during the tenure of the claim then the payoff of the claim is valued at a fraction of the default-free claim. Consequently, the payoff of the claim in the firm insolvency state can be expressed as:

$$\bar{P}(T, T) = P(T, T) \frac{V(T, T)}{D(T, T)}$$

Notice, that  $\frac{V(T, T)}{D(T, T)}$  denotes the fraction of the default-free contingent claim  $P(T, T)$  paid to the claim-holder and is also referred to as the *recovery rate*. In the Merton (1974)

[94] framework default can only occur at maturity  $T$  and combining the binary states of nature of solvency and insolvency of the firm into a single expression we can specify the payoff of a credit risky claim as:

$$\bar{P}(T, T) = \begin{cases} P(T, T) & \text{if } V_T \geq D_T \\ \omega(T, T)P(T, T) & \text{if } V_T < D_T \end{cases}$$

where  $\omega(t)$  is the recovery rate. We define the non-defaultable money market account as:

$$B(t, T) = \exp \left( \int_t^T r_u du \right)$$

to be the numéraire asset. On the basis of the assumptions given above we state that in the risk-neutral economy and through the direct application of risk-neutral valuation techniques the price of a credit risky contingent claim can be expressed as:

$$\bar{P}(t, T) = B_t \mathbb{E}_{\mathbb{Q}} \left[ B_T^{-1} P(T, T) (I_{\{V_T \geq D_T\}} + \omega_T I_{\{V_T < D_T\}}) \middle| \mathcal{F}_t \right] \quad (6.14)$$

Clearly,  $\bar{P}(T, T)$  is an arbitrary claim and can, for example, admit derivatives with stochastic payoffs. In our model we specify that  $\bar{P}(T, T)$  represents a credit risky put option.

#### 6.6.4 Closed Form Formula for a Credit Risky Put Option

The theory for pricing vanilla European default-free contingent claims in the Black-Scholes (1973) [20] risk-neutral valuation framework can be extended to the pricing of credit risky contingent claims. To evaluate an arbitrary credit risky contingent claim  $\bar{P}(T, T)$  required the modelling of the associated variables  $V_t$ , the firm value process, and  $S_t$ , the share price process. We are now in a position to illustrate the primary aim of this section, that is, to derive a pricing formula for a credit risky put option. In general the payoff of a put option is defined as:

$$P(T) = (X - S_T)^+ \quad (6.15)$$

where  $S_T$  is the price of the underlying security at maturity  $T$  and  $X$  the strike price of the option. Formally, the price of a credit risky put option is given in the following proposition that appears in Ammann (2001) [4] albeit without a formal derivation. We show a proof for the option formula in Appendix 6.

**Proposition 1:** The price  $\bar{P}_t$  of a credit risky option with guaranteed payoff  $\bar{P}_T = (X - S_T)^+$  and payoff  $\bar{P}_T = \omega(X - S_T)^+$ , with  $\omega = \frac{V_T}{D}$  in the case of one sided counterparty default, is given by:

$$\begin{aligned} \bar{P}_t &= e^{-r(T-t)} X N_2(-b_1, b_2, \nu) - S_t N_2(-a_1, a_2, \nu) \\ &+ \frac{V_T}{D} \left( X N_2(-d_1, d_2, -\nu) - e^{(r+\nu\sigma_S\sigma_V)(T-t)} S_t N_2(-c_1, c_2, -\nu) \right) \end{aligned}$$

with parameters shown in Appendix 6.  $N_2(\cdot)$  denotes the cumulative distribution function of a bivariate standard joint normal random variable. Although the specification of the interest rates,  $r$ , and the instantaneous standard deviation of the returns,  $\sigma_V$  and  $\sigma_S$ , suggest that these variables are constants it is sufficient that they be deterministic.

### 6.6.5 Numerical Analysis

In this section we attempt to analyze the parameters of the model through the valuation of a credit risky put option. Where applicable comparisons will be made with the characteristics of a risk-free Black-Scholes (1973) [20] option. We choose a base case example represented by a highly leveraged corporate with 90% debt-to-asset ratio issuing an *at the money* put option with two years to maturity and there is no correlation between the returns of the counterparty assets and the return on the asset underlying the option.

The value of this credit risky put option is required to satisfy the PDE given by expression (6.13) and is solved analytically given the particular boundary conditions. The solution to this PDE is shown in Proposition 1. The following nine factors affecting the credit risky put option will be examined:

1. the current value of the share price ( $S$ )
2. the strike price ( $X$ )
3. the volatility of the share price ( $\sigma_S$ )
4. the risk-free interest rate ( $r$ )
5. the time to expiration ( $T - t$ )
6. the current value of the firm's assets ( $V$ )
7. the volatility of the firm's assets ( $\sigma_V$ )
8. the counterparty's liabilities ( $D$ )
9. correlation ( $\rho$ ) between  $S$  and  $V$

The exact parameters for the base case example are the value of the firm  $V = 200$ , the strike price  $X = 100$ , the share price  $S = 100$ , the value of debt  $D = 180$ , the volatility of the firm's assets  $\sigma_V = 0.2$ , the volatility of the share price  $\sigma_S = 0.2$ , the risk-free interest rate  $r = 0.05$  and the correlation between  $S$  and  $V$  is  $\rho = 0$ . The numerical results is stored in the hardware accompanying this dissertation.

In Table 6.4 we present values of credit risky put options based on the model of this section for the base case example at varying recovery rate levels. In addition, we show option values for sensitivities to changes in the base case parameter values. In the last column of the table we show the Black-Scholes (1973) [20] option prices corresponding to

each case of change in sensitivity of parameter values. We use the approximation of the bivariate normal distribution from West (2004) [120].

Case		$\omega = 0.5$	$\omega = 0.75$	$\omega = 1.0$	$\omega = 5.0$	<i>BS Value</i>
Base Case		0.91	1.36	1.69	1.77	1.77
$\sigma_S$	0.1	0.35	0.53	0.66	0.69	0.69
$\sigma_S$	0.3	1.47	2.20	2.74	2.87	2.87
$\sigma_V$	0.1	0.91	1.36	1.74	1.77	1.77
$\sigma_V$	0.3	0.91	1.34	1.64	1.77	1.77
$T - t$	0.3	0.74	1.11	1.40	1.45	1.45
$T - t$	0.75	1.06	1.58	1.93	2.04	2.04
$X$	35	0.18	0.26	0.33	0.34	0.34
$X$	50	4.61	6.90	8.59	8.99	8.99
$S_t$	35	2.40	3.59	4.46	4.68	4.68
$S_t$	50	0.05	0.08	0.09	0.10	0.10
$\rho$	-0.5	0.99	1.48	1.66	1.77	1.77
$\rho$	0.5	0.83	1.23	1.54	1.77	1.77

Table 6.4: Summary of Values of Credit Risky Put Options

A natural consequence of higher default firm values is higher rates of recovery. We observe that for the base case example that higher rates of recovery correspond to an increase in price for credit risky options. Notably, for a firm value equal to the payoff of the option at maturity the risky option is marginally cheaper than the price of its risk-less counterpart. However, for a firm value that is a multiple of 5 times its debt value the price of credit risky and risk-less options are identical. Clearly, the market considers an option writer with a high assets-to-debt ratio to have a negligible risk of default. This trend of higher rates of recovery corresponding to an increase in price of credit risky put options is similarly observed at varying sensitivities of option parameters. While this result exactly confirms our intuition we find it instructive to analyze each parameter with regard to credit risky put option prices. We show this analysis in the examples that follow. The data used to plot the figures is given in Table 6.5 in Appendix 6.

#### 6.6.5.1 Strike Price ( $X$ ) and Share Price ( $S$ )

The payoff of a European put option is the amount by which the *strike price*  $X$  exceeds the *share price*  $S$ . Thus as the share price decreases or the strike price increases the

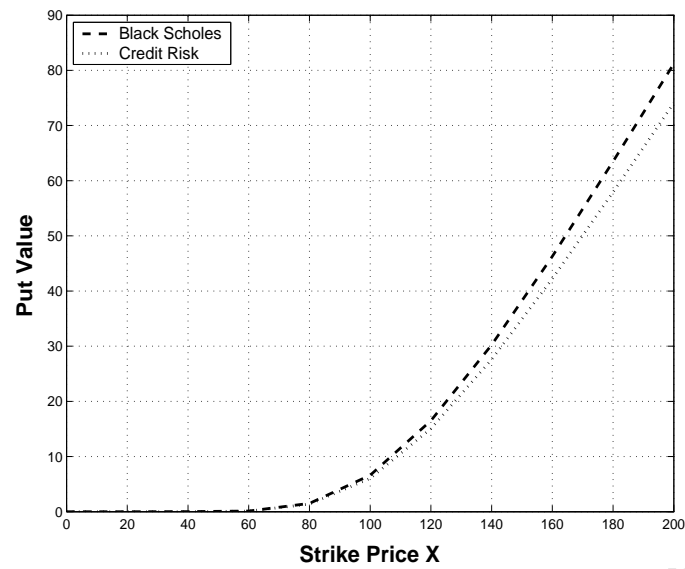


Figure 6.5: Credit Risky Put expressed as a function of the Strike Price X

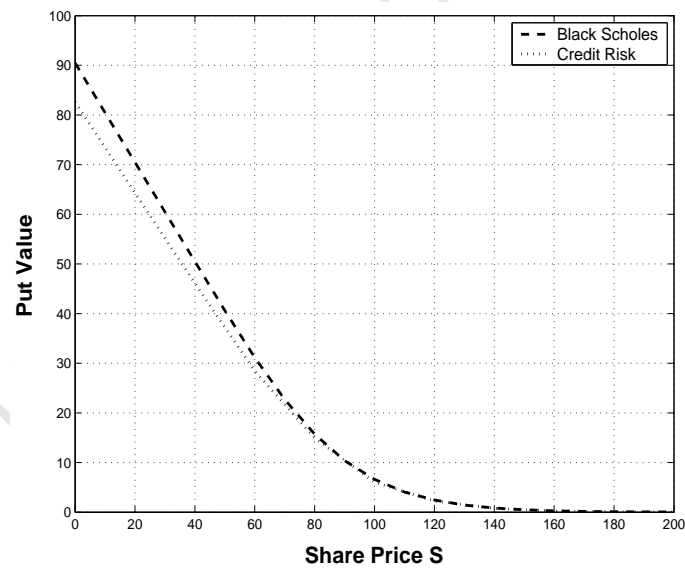


Figure 6.6: Credit Risky Put expressed as a function of the Share Price S



expected payoff will increase and the credit risky put option will subsequently increase in value as shown in Figures 6.5 and 6.6. In addition, the *delta* of a risk-less Black-Scholes (1973) [20] put option converges to -1 (see spreadsheet *delta* stored in accompanying hardware) as the option moves *deep in the money* and close to time to expiry as there is an increased likelihood of the option being exercised. In this model setup we observe an identical outcome for a credit risky put option, that is, the delta of the put option converges to -1 as the share price goes to zero, it is more likely that the put option will end up *in the money*. By implication the put option is most valuable at this point so that the position in the share should be near maximum as well. On the other hand, as expected the cost of a credit risky put option is always cheaper than the Black-Scholes (1973) [20] put option.

#### 6.6.5.2 Share Price Volatility ( $\sigma_S$ )

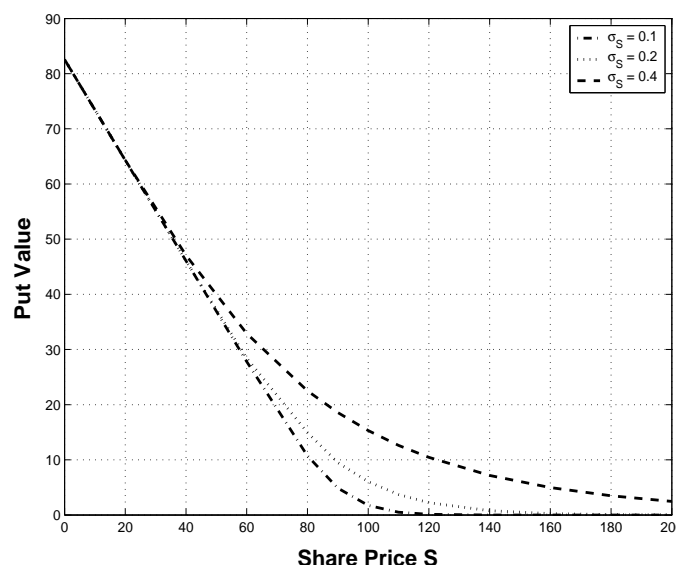


Figure 6.7: The effect of the Share Price Volatility on the value of a Credit Risky Put

We concern ourselves with the volatility of the share price which is just a measure of the risk of future share price movements. Traditionally, there are two approaches to estimating volatility in option pricing models, *historical* and *implied* volatility. To be consistent with the Black-Scholes (1973) [20] model assumptions we choose the share price volatility to be a constant value. The implication of increases in volatility is larger variances in share prices. As a result a portfolio holding this share will reflect relative increases or decreases in value. Further, the holder of a risk-less option can gain from increases in volatility of the share price since options have limited downside risks. However, for a holder of a put option there is both bounded gains and losses with a maximum gain of the strike price at zero share price and with the loss of the option premium at share prices larger than the strike price. Thus as the volatility increases option prices tend to increase.

Similarly, from Figure 6.7 we observe that for a credit risky put the value of the option increases with volatility of share prices. Also, option prices are identical for varying volatilities up until share prices of 30 but thereafter uniformly diverge with the largest volatility  $\sigma_S = 0.4$  producing the most expensive option prices. A potential explanation for identical option prices at varying volatilities is that *deep in the money* options at close to time to expiry will certainly be exercised, *ceterus paribus*. On the other hand, while the payoffs on these options are contingent liabilities on the counterparty's balance sheet the increased likelihood on the options being exercised can increase the risk of default of the counterparty if, for example, short-term cashflows are unpredictable, thus these option prices bear the same level of risk and have identical prices. In addition, our intuition is exact in so far as for low volatility share prices, for example, at  $\sigma_S = 0.1$  the variance of the share prices are smaller hence the option prices are cheaper and quickly approach zero for *out of the money* options. This effect is clearly demonstrated for higher volatility share prices where option prices tend to increase with volatility but reflect a much slower speed of decay towards zero for *out of the money* options. At each level of volatility option prices show maximum variance in the range  $X \pm 40$  where  $X = 100$  is the strike price. This is a plausible expectation since with sufficient time to expiry share prices can either decrease for the put option to expire *in the money* or increase for the put option to expire *out of the money* albeit option prices in the range  $X + 40$  are cheaper than prices in the range of  $X - 40$ .

### 6.6.5.3 Risk-free Interest Rate ( $r$ )

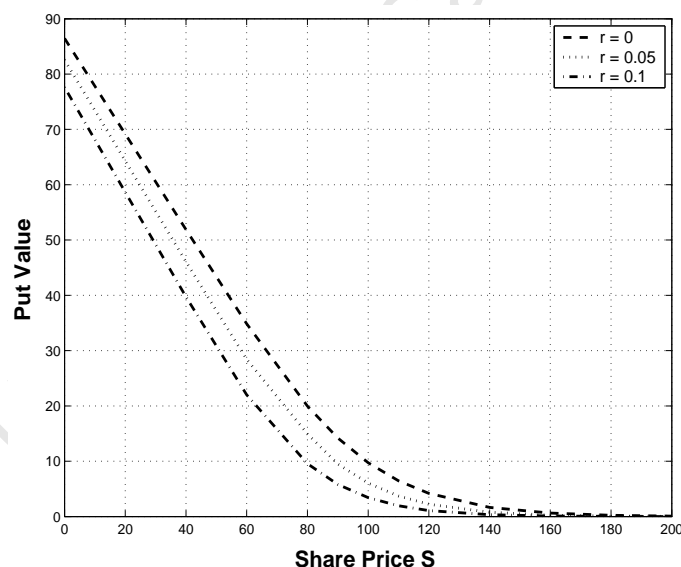


Figure 6.8: The effect of the Risk-free interest rate on the value of a Credit Risky Put

In the model setup we introduce the risk-neutral probability measure  $\mathbb{Q}$  that trans-

lates the drift component of the firm value process  $V$  and the share price process  $S$  to the risk-free interest rate  $r$ . Consider the case where a rising economy is correlated with rising interest rates. This means that the expected growth of both the firm value and share price will appreciate accordingly. The empirical implication is that an increasing firm value indicates an increasing likelihood that the counterparty (option writer or obligor) will meet the payoff obligation on the option. As a result the value of the option increases. On the other hand, while an increase in the share price may positively affect the payoff of an *in the money* option this payoff is also a contingent liability for the counterparty that can impact to increase the risk of default at the option maturity.

From Figure 6.8 we observe that at the varying levels of interest rates each curve is left upward sloping and appear almost parallel to each other but at  $X = 40$  the curves begin to gradually converge together. This phenomenon is expected since at  $X > 100$  the put options move *out of the money* and at close to time to expiry the options are invariant to the varying levels of the risk-free interest rate as they are likely to expire without being exercised.

#### 6.6.5.4 Time to Expiration ( $T$ )

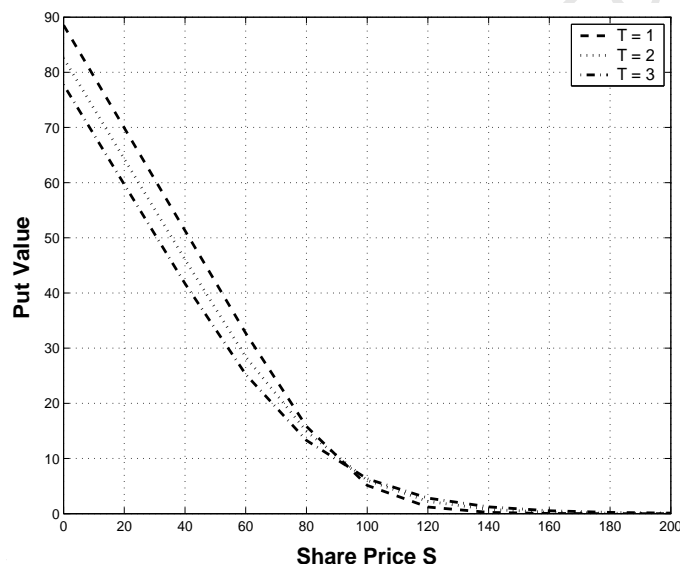


Figure 6.9: The effect of the Expiration date on the value of a Credit Risky Put

For a Black-Scholes (1973) [20] type option an increase in the time to expiry can have an ambiguous effect on an European put option. First, since the share price is modelled as a diffusion process with a positive drift, this implies an increasing term structure of share prices and hence the associated long run variance of expected share returns increases. As a result the put option becomes more valuable. Second, as time to maturity increases the discounted value of the exercise price is cheaper. This makes the put option less valuable.

For a credit risky put option, if the effect of higher variances of expected share returns dominates this implies an increased expected payoff that can subsequently increase the likelihood of the counterparty defaulting on the option payoff at maturity. On the other hand, the firm value appreciates at the identical rate of return such that it has increased its capacity to meet contingent obligations.

In Figure 6.9 we show the relationship between put and share prices as affected by varying levels of time to maturity. We observe that a credit risky put option is most valuable at  $T = 1$ , the shortest time to maturity, in the range 0 to  $X - 10$ , where  $X$  is the strike price. Additionally, in this range the option is *deep in the money*. This occurrence can be attributed to the dominant effect of short time to expiry on the present value of the exercise price makes the option more valuable. However, in the range  $S > X - 10$  we observe a crossover such that at time to maturity  $T = 1$  option prices are the cheapest. This also approximates the *out of the money* range of the option. The dominant effect in this range could be the increasing variance in share price returns together with a longer time to maturity makes the put option more valuable relative to shorter time to expiry options. While this maybe the case the options are also *deep out of the money* which makes them less likely to be exercised that reduces the risk of counterparty default.

#### 6.6.5.5 Counterparty Assets ( $V$ )

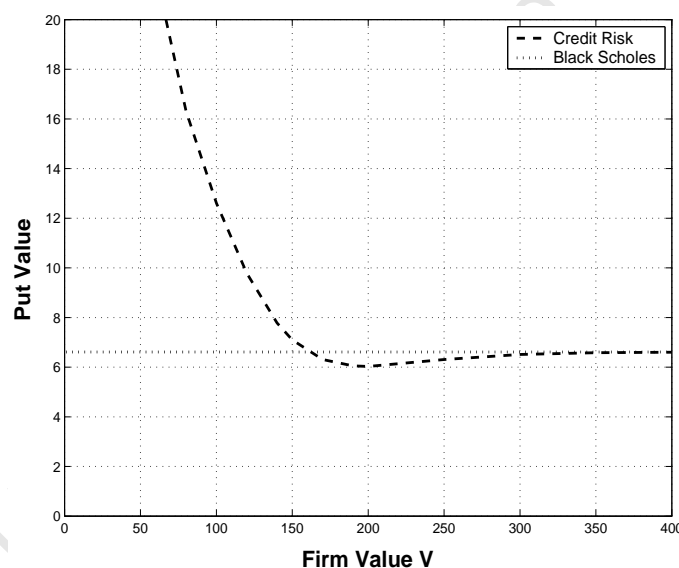


Figure 6.10: At-the-Money Credit Risky Put option as a function of the Counterparty Asset Value,  $V$

In this structural model framework we follow the Merton (1974) [94] firm value approach where default can only occur when the counterparty assets is in breach of the default threshold at option maturity. The firm value is modelled as a diffusion process

with positive drift and together with a higher initial value of firm assets there is an increased probability that the firm value may drift away from the default threshold which decreases the likelihood that it will default on the contingent payoff at the option maturity. At increasing levels of firm value the probability of default declines toward zero the value of the credit risky put option asymptotically tend towards the value of the risk-less option as shown in Figure 6.10 for an *at the money* put option and in Figure 6.11 for a *deep in the money* put option.

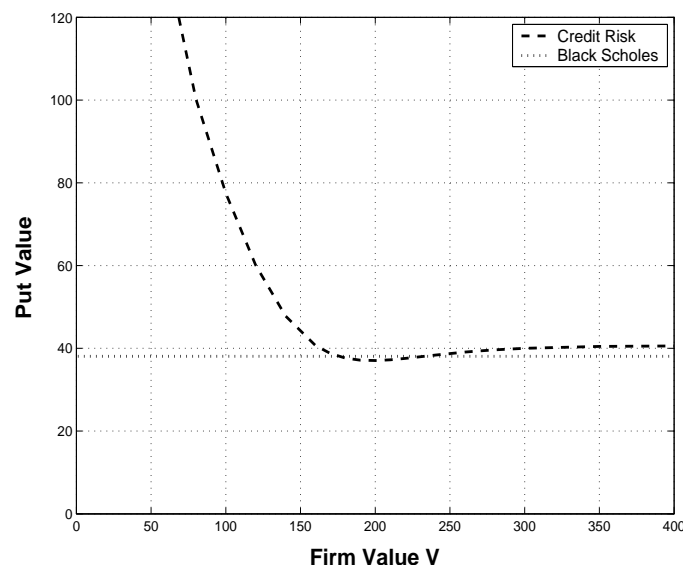


Figure 6.11: Deep in-the-Money Credit Risky Put option as a function of the Option Writer's Asset Value,  $V$

In Figures 6.10 and 6.11 for an *at the money* and *deep in the money* put there is a sharp decline in value as the firm value increases from 160 to 400. For firm values in the range 0 to 160 the counterparty probability of default is 1 since the firm value is modelled as a diffusion process and cannot jump in the next instant to a value larger than the default threshold of 200. Intuitively, we know that the nominal payoff of an *at the money* option or when the firm is in default is zero. However, for both *at the money* and *deep in the money* options at default firm values the puts have a high option value. This effect for *in the money* options can be explained by the fact the option has a positive nominal net value. In addition, the options could have long dated times to expiry that gives firm sufficient time for its assets to appreciate above its default value. As shown the figure for a *deep in the money* option at firm values  $V \geq 220$  the credit risky option is more expensive than the Black-Scholes (1973) [20] option. An implication of this occurrence is that a *deep in the money* option is a senior claim relative to an *at the money* option in the event of bankruptcy of the counterparty.

### 6.6.5.6 Asset Volatility ( $\sigma_V$ )

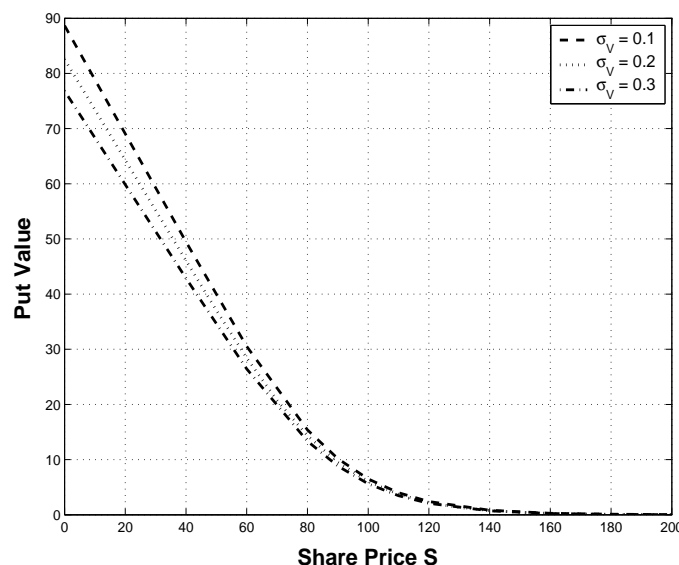


Figure 6.12: The effect of the Asset Volatility on the value of a Credit Risky Put

Generally, in option pricing we can observe the value of the underlying asset and hence calculate its implied volatility by inverting the option valuation formula. In contrast, the evolution of asset value of the firm is not an easily observable process. As such in structural models we commonly rely on estimates or indirect methods to evaluate the asset volatility, for example, recall that equity is a call option on the firm value. Notwithstanding, that a risk-less put option has limited upside gain from an increase in volatility of the share price similarly a credit risky put option has a bounded upside gain from an increase in the volatility of the counterparty's assets. In other words, if the asset price process drift upwards with an increase in asset value the variance of the asset returns bear a lesser gain to the option holder than if the asset price process drifts downward with a decrease in asset value that bears a larger probability of default of the counterparty. From Figure 6.12 we observe that the option priced with the lowest asset volatility is most valuable and credit risky put options that are *out of the money* are invariant to asset volatility dynamics.

### 6.6.5.7 Counterparty Debt ( $D$ )

We intuitively assert that as the levels of debt of the counterparty increases, *ceterus paribus*, implies an increased risk of default on a potential option payoff at maturity. In addition, this impacts on the recovery rate with lower levels of firm value to be distributed to the holder of the contingent claim. Moreover, as the debt levels of the counterparty grows larger the value of the put option ought to decrease. However, from Figure 6.13 we observe that there is negligible difference in the value of the options for increasing counterparty debt levels. The implication of this phenomenon is that the market implicitly

determines a threshold debt level and one that is much lower than the current firm value for counterparty default and associates a marginal risk of default for higher levels of debt.

#### 6.6.5.8 Correlation ( $\rho$ ) between Share Price and Firm Value

The underlying asset diffusion processes for the share price and firm value are correlated through the Brownian motions  $\tilde{B}_S$  and  $\tilde{B}_V$ . A firm with a relatively high and non-negative correlation coefficient may be inclined to write options on its underlying share price. Potential option holders may view this favorably as a rising trend in the share price is associated with an increasing firm value. As a result, the counterparty will be deemed more likely to meet its contingent payoff obligations. On the other hand, for a firm that has a relatively high and negative correlation coefficient a firm's put options may be deemed particularly valuable as a decreasing share price is associated with increasing firm value. This implies that we have a higher intrinsic value of the put option with a higher likelihood that the company will meet its contingent payoff obligations. As shown in Figure 6.14 *in the money* put options with a high negative correlation are relatively more valuable than options written on correlations that are zero or positive. Also, we observe that this trend is reversed for *out of the money* put options, that is, options are cheaper for high and negative correlations when share prices are increasing with an associated decrease in firm value.

#### 6.6.6 Supplementary Remarks

In this section we postulated a non-complex credit risk model based on the traditional Merton (1974) [94] firm value model. The model assumptions were similar to that proposed for the Black-Scholes (1973) [20] framework with the additional assumption of one sided counterparty default risk. We stated an explicit pricing formula for a credit risky put option and derived a closed form solution for the option price with the assumption that the value of the firm and the share price process follow a standard geometric Brownian motion. The model is sufficiently flexible to be adapted to various combinations of deterministic, and stochastic interest rates and liabilities, respectively, and underscores its primary utility of facilitating closed form solutions for option prices under these variations.

### 6.7 Summary

The market for credit risk derivatives has grown phenomenally and has become an important type of financial instrument in the broader arena of the over-the-counter derivatives market. The special interest in the credit risk derivative market is twofold. First, banks and other obligors are able to raise wide margins on exotic financial instruments that are priced off difficult to model phenomena. Second, the advent of credit risk derivatives has advanced the completion of financial markets both by creating arbitrage opportunities on increasing efficient pricing methodologies and by allowing corporates and other

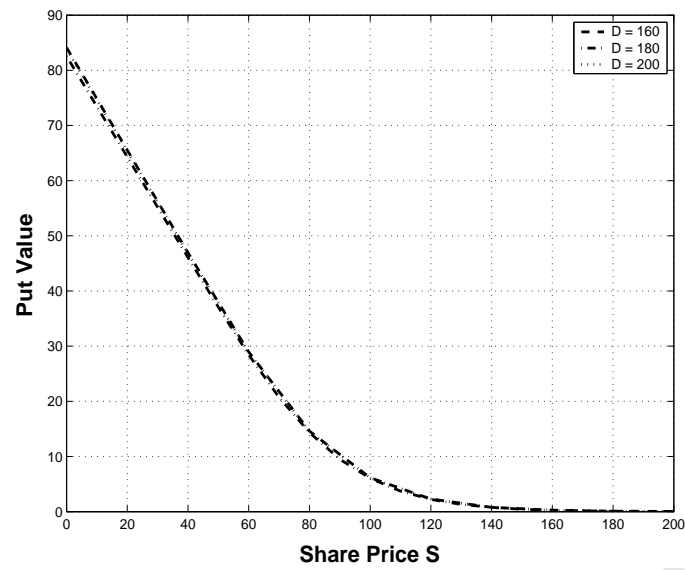


Figure 6.13: The effect of Counterparty Debt on the value of a Credit Risky Put

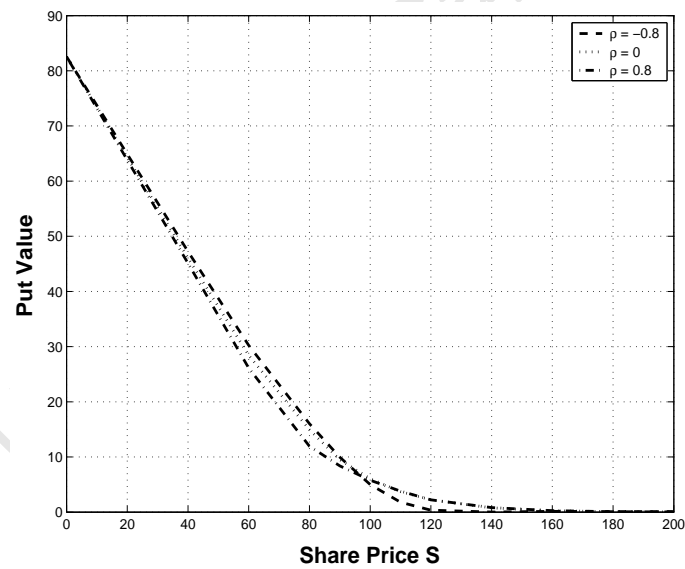


Figure 6.14: The effect of the Correlation Coefficient  $\rho$  between  $S$  and  $V$  on the value of a Credit Risky Put



obligors new structured products to manage their credit risk exposures.

First, we described three basic instruments to structure a credit risk derivative: (1) the total return swap, is a structure where we link a stream of payments to the total return on a pre-defined asset. Here the risk exposure of the underlying asset is transferred through the total return payment stream to the total return receiver, (2) a credit default swap, is designed such that the counterparty payer serves as a insurer and receives the credit risk associated with a specified credit event. The payoff on a credit default swap is contingent on the credit event, for example, a credit rating downgrade or a coupon payment default, and (3) a credit spread option, is a structure where the obligor of this derivative serves as an insurer and receives the credit risk associated with the credit quality of the underlying asset decreasing and hence triggers the linked credit spread to widen. The payoff of the credit spread is typically based on a loan obligation or bond yield. Typically, market makers trading with credit risk would prefer to deal with credit risk derivatives than with traditional lending and borrowing instruments.

In the following section we produced an overview of the Das and Sundaram (2000) [34] paper that offered a compact approach to arbitrage free pricing of credit risk derivatives based on the reduced form model framework. They present a modelling approach that can be implemented with ease and makes use of market observables as inputs. As the first step, the model postulates input parameters as the term structure of risk free forward rates and the term structure of observable credit spreads. Next, the associated term structures of the diffusion parameters of the forward rates and spreads, respectively, expanded the development of an arbitrage free double-binomial lattice using the discrete time HJM framework. The logit equation serves as the useful mechanism to decompose values at each node on the lattice into default probabilities and recovery rates. The computer implementation uses a multi-dimensional recursive equation system that seamlessly processes forward induction and backward recursion consistent with the absence of arbitrage. Although the model is arbitrage free, its encompasses path dependence and handles a variety of credit default financial instruments it also underpins a broad scope for additional research. Finally, as an application of the arbitrage free credit risk derivative model we use a computer implementation of the model to price a credit spread option.

The final section of this chapter postulates a credit risk derivative model based on the classic Merton (1974) [94] firm value model methodology. We make the assumption of one sided counterparty default risk in this structural model framework. As an application of the firm value credit risk derivative model we price a credit risky put option with the additional assumption of deterministic interest rates and deterministic liabilities. We conclude by deriving a closed form pricing formula for the credit risky put option.

## 6.8 Appendix 6

In this section we show the parameters and the proof of Proposition 1.

### 5.1 Parameters as shown in Proposition 1

$$\begin{aligned}
 a_1 &= \frac{\ln \frac{S_t}{X} + (r + \frac{1}{2}\sigma_S^2)(T-t)}{\sigma_S\sqrt{T-t}} \\
 a_2 &= \frac{\ln \frac{V_t}{D} + (r - \frac{1}{2}\sigma_V^2 + \nu\sigma_S\sigma_V)(T-t)}{\sigma_V\sqrt{T-t}} \\
 b_1 &= \frac{\ln \frac{S_t}{X} + (r - \frac{1}{2}\sigma_S^2)(T-t)}{\sigma_S\sqrt{T-t}} = a_1 - \sigma_S\sqrt{T-t} \\
 b_2 &= \frac{\ln \frac{V_t}{D} + (r - \frac{1}{2}\sigma_V^2)(T-t)}{\sigma_V\sqrt{T-t}} = a_2 - \nu\sigma_S\sqrt{T-t} \\
 c_1 &= \frac{\ln \frac{S_t}{X} + (r + \frac{1}{2}\sigma_S^2 + \nu\sigma_S\sigma_V)(T-t)}{\sigma_S\sqrt{T-t}} = a_1 + \nu\sigma_V\sqrt{T-t} \\
 c_2 &= -\frac{\ln \frac{V_t}{D} + (r + \frac{1}{2}\sigma_V^2 + \nu\sigma_S\sigma_V)(T-t)}{\sigma_V\sqrt{T-t}} = -a_2 - \nu\sigma_S\sqrt{T-t} \\
 d_1 &= \frac{\ln \frac{S_t}{X} + (r - \frac{1}{2}\sigma_S^2 + \nu\sigma_S\sigma_V)(T-t)}{\sigma_S\sqrt{T-t}} = a_1 + (\nu\sigma_V + \sigma_S)\sqrt{T-t} \\
 d_2 &= -\frac{\ln \frac{V_t}{D} + (r + \frac{1}{2}\sigma_V^2)(T-t)}{\sigma_V\sqrt{T-t}} = -(a_2 - (\nu\sigma_S + \sigma_V)\sqrt{T-t})
 \end{aligned}$$

### 5.2 Proof of Proposition 1

From (6.14) and (6.15) the price of a credit risky put option can be expressed as:

$$\bar{P}_t = B_t \mathbb{E}_{\mathbb{Q}} \left[ B_T^{-1} (X - S_T)^+ (I_{\{V_T \geq D\}} + \omega_T I_{\{V_T < D\}}) \middle| \mathcal{F}_t \right]$$

with  $\omega_T = \frac{V_T}{D}$ .  $\mathbb{Q}$  denotes the risk-neutral measure and  $B_t$  the money market account. The expectation can be expanded into several terms such that

$$\bar{P}_t = \bar{P}_1 - \bar{P}_2 + \bar{P}_3 - \bar{P}_4$$

and the individual terms are expressed as:

$$\begin{aligned}
\bar{P}_1 &= B_t \mathbb{E}_{\mathbb{Q}} \left[ B_T^{-1} X I_{\{X > S_T\}} I_{\{V_T \geq D\}} \middle| \mathcal{F}_t \right] \\
\bar{P}_2 &= B_t \mathbb{E}_{\mathbb{Q}} \left[ B_T^{-1} S_T I_{\{X > S_T\}} I_{\{V_T \geq D\}} \middle| \mathcal{F}_t \right] \\
\bar{P}_3 &= B_t \mathbb{E}_{\mathbb{Q}} \left[ B_T^{-1} X \omega_T I_{\{X > S_T\}} I_{\{V_T < D\}} \middle| \mathcal{F}_t \right] \\
\bar{P}_4 &= B_t \mathbb{E}_{\mathbb{Q}} \left[ B_T^{-1} S_T \omega_T I_{\{X > S_T\}} I_{\{V_T < D\}} \middle| \mathcal{F}_t \right]
\end{aligned} \tag{6.16}$$

We can evaluate each of the four terms separately. If we assume that  $B_t \mathbb{E}_{\mathbb{Q}}[B_T^{-1} | \mathcal{F}_t] = e^{-r(T-t)}$  then we can obtain closed form solutions.

**Evaluation of term  $\bar{P}_2$ .** Substituting for  $S_T$  gives:

$$\bar{P}_2 = \mathbb{E}_{\mathbb{Q}} \left[ S_t e^{-\frac{1}{2}\sigma_S^2(T-t) + \sigma_S(\tilde{B}_T - \tilde{B}_t)} I_{\{X > S_T\}} I_{\{V_T \geq D\}} \middle| \mathcal{F}_t \right]$$

Choose  $\tilde{y} = \frac{\tilde{B}_T - \tilde{B}_t}{\sqrt{T-t}}$  such that its probability distribution is  $N(0,1)$ .  $\bar{P}_2$  can be expressed in terms of a bivariate normal distribution

$$\begin{aligned}
\bar{P}_2 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S_t e^{-\frac{1}{2}\sigma_S^2(T-t) + \sigma_S \tilde{y}_1 \sqrt{T-t}} I_{\{X > S_T\}} I_{\{V_T \geq D\}} \\
&\quad \frac{1}{2\pi\sqrt{1-\nu^2}} e^{\left(-\frac{1}{2(1-\nu^2)}(\tilde{y}_1^2 - 2\nu\tilde{y}_1\tilde{y}_2 + \tilde{y}_2^2)\right)} d\tilde{y}_1 d\tilde{y}_2
\end{aligned} \tag{6.17}$$

to re-arrange terms we shall use the following equality:

$$\begin{aligned}
&\frac{\tilde{y}_1^2 - 2\nu\tilde{y}_1\tilde{y}_2 + \tilde{y}_2^2}{2(1-\nu^2)} + p\tilde{y}_1 + q\tilde{y}_2 + r \\
&= \frac{-(\tilde{y}_1 - p - \nu q)^2 - 2\nu(\tilde{y}_1 - p - \nu q)(\tilde{y}_2 - q - \nu p) + (\tilde{y}_2 - p - \nu q)^2}{2(1-\nu^2)} \\
&\quad + \frac{1}{2}p^2 + \nu pq + \frac{1}{2}q^2 + r
\end{aligned} \tag{6.18}$$

Substituting (6.18) into  $\bar{P}_2$  we get:

$$\begin{aligned}
\bar{P}_2 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S_t I_{\{X > S_T\}} I_{\{V_T \geq D\}} \\
&\quad \frac{1}{2\pi\sqrt{1-\nu^2}} e^{\left(-\frac{1}{2(1-\nu^2)}(u_1^2 - 2\nu u_1 u_2 + u_2^2)\right)} d\tilde{y}_1 d\tilde{y}_2
\end{aligned} \tag{6.19}$$

with  $u_1 = \tilde{y}_1 - \sigma_S \sqrt{T-t}$  and  $u_2 = \tilde{y}_2 - \sigma_S \sqrt{T-t}$ .

We introduce an equivalent probability measure defined by

$$\frac{d\dot{\mathbb{Q}}}{d\mathbb{Q}} = \exp\left(\delta \tilde{B}_T - \frac{1}{2}\sigma^2 T\right) \quad (6.20)$$

with  $\delta$  and  $\tilde{B}$  vectors in  $\mathbb{R}^2$ .  $\delta$  is defined to have elements  $\delta_S = \sigma_S$  and  $\delta_V = \nu\sigma_S$ .

By Girsanov's theorem  $\dot{B}_t = \tilde{B}_t - \delta t$  is a  $\mathbb{R}^2$ -valued standard Brownian motion under  $\dot{\mathbb{Q}}$ . Therefore,

$$\tilde{y} = \frac{\tilde{B}_T - \tilde{B}_t}{\sqrt{T-t}} = \frac{\dot{B}_T - \dot{B}_t + \delta(T-t)}{\sqrt{T-t}} = \dot{y} + \delta\sqrt{T-t} \quad (6.21)$$

Under the equivalent martingale measure  $\dot{\mathbb{Q}}$  the density in expression (6.17) is a standard bivariate normal distribution.  $\bar{P}_2 = S_t N_2(-a_1, a_2, \nu)$  with the parameters  $a_1$  and  $a_2$  of the bivariate joint distribution function  $N$  to be determined by the evaluation of the indicator functions. The indicator functions can be evaluated as:

$$\begin{aligned} \mathbb{E}_{\dot{\mathbb{Q}}}[I_{\{X > S_T\}}] &= \dot{\mathbb{Q}}(X > S_T) \\ &= \dot{\mathbb{Q}}\left(X > S_t e^{(r - \frac{1}{2}\sigma_S^2)(T-t) + \sigma_S(\dot{B}_T - \dot{B}_t + \sigma_S(T-t))}\right) \\ &= \dot{\mathbb{Q}}\left(\ln X - \ln S_t - \left(r + \frac{1}{2}\sigma_S^2\right)(T-t) > \sigma_S(\dot{B}_T - \dot{B}_t)\right) \\ &= \dot{\mathbb{Q}}\left(\dot{y}_1 < -\frac{\ln S_t - \ln X + \left(r + \frac{1}{2}\sigma_S^2\right)(T-t)}{\sigma_S\sqrt{T-t}}\right) \end{aligned} \quad (6.22)$$

$$\begin{aligned} \mathbb{E}_{\dot{\mathbb{Q}}}[I_{\{V_T \geq D\}}] &= \dot{\mathbb{Q}}(V_T > D) \\ &= \dot{\mathbb{Q}}\left(V_t e^{(r - \frac{1}{2}\sigma_V^2)(T-t) + \sigma_V(\dot{B}_T - \dot{B}_t + \nu\sigma_S(T-t))} > D\right) \\ &= \dot{\mathbb{Q}}\left(\sigma_V(\dot{B}_T - \dot{B}_t) > \ln D - \ln V_t - \left(r - \frac{1}{2}\sigma_V^2 + \nu\sigma_S\sigma_V\right)(T-t)\right) \\ &= \dot{\mathbb{Q}}\left(\dot{y}_2 < \frac{\ln V_t - \ln D + \left(r - \frac{1}{2}\sigma_V^2 + \nu\sigma_S\sigma_V\right)(T-t)}{\sigma_V\sqrt{T-t}}\right) \end{aligned} \quad (6.23)$$

Now we consider the properties of the joint normal distribution to get:

$$\begin{aligned}
& \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I_{\{y_1 \geq a_1\}} I_{\{y_2 \geq a_2\}} f(y_1, y_2, \nu) dy_2 dy_1 \\
&= \int_{a_1}^{\infty} \int_{a_2}^{\infty} f(y_1, y_2, \nu) dy_2 dy_1 \\
&= \int_{a_1}^{\infty} \int_{-\infty}^{-a_2} f(y_1, y_2, -\nu) dy_2 dy_1 \\
&= \int_{-\infty}^{-a_1} \int_{-\infty}^{-a_2} f(y_1, y_2, \nu) dy_2 dy_1
\end{aligned} \tag{6.24}$$

where  $f(y_1, y_2, \nu)$  denotes the joint density function. From the evaluation of the indicator functions in expressions (6.22) and (6.23) and from the equalities in expression (6.24), it follows that

$$\bar{P}_2 = S_t N_2(-a_1, a_2, \nu)$$

where the parameters of the distribution function are given by

$$\begin{aligned}
a_1 &= -\frac{\ln \frac{S_t}{X} + (r + \frac{1}{2}\sigma_S^2)(T-t)}{\sigma_S \sqrt{T-t}} \\
a_2 &= \frac{\ln \frac{V_t}{D} + (r - \frac{1}{2}\sigma_V^2 + \nu\sigma_S\sigma_V)(T-t)}{\sigma_V \sqrt{T-t}}
\end{aligned}$$

**Evaluation of term  $\bar{P}_1$ :** The first term can be written as

$$\begin{aligned}
\bar{P}_1 &= B_t \mathbb{E}_{\mathbb{Q}} [B_T^{-1} X I_{\{X > S_T\}} I_{\{V_T > D\}} | \mathcal{F}_t] \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X e^{-r(T-t)} I_{\{X > S_T\}} I_{\{V_T > D\}} \\
&\quad \frac{1}{2\pi\sqrt{1-\nu^2}} e^{\left(-\frac{1}{2(1-\nu^2)}(\tilde{y}_1^2 - 2\nu\tilde{y}_1\tilde{y}_2 + \tilde{y}_2^2)\right)} d\tilde{y}_1 d\tilde{y}_2
\end{aligned}$$

From (6.24) and since  $e^{-r(T-t)}X$  is a constant the expression for  $\bar{P}_1$  becomes

$$\bar{P}_1 = X e^{-r(T-t)} N_2(-b_1, b_2, \nu)$$

where  $b_1$  and  $b_2$  are again determined by the evaluation of the indicator functions. In this case the indicator functions can be evaluated without a change of measure and we get:

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[I_{\{X > S_T\}}] &= \mathbb{Q}(X > S_T) \\
&= \mathbb{Q}\left(X > S_t e^{(r - \frac{1}{2}\sigma_S^2)(T-t) + \sigma_S(\tilde{B}_T - \tilde{B}_t)}\right) \\
&= \mathbb{Q}\left(\ln X - \ln S_t - \left(r - \frac{1}{2}\sigma_S^2\right)(T-t) > \sigma_S(\tilde{B}_T - \tilde{B}_t)\right) \\
&= \mathbb{Q}\left(\tilde{y}_1 < -\frac{\ln \frac{S_t}{X} + (r - \frac{1}{2}\sigma_S^2)(T-t)}{\sigma_S\sqrt{T-t}}\right)
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[I_{\{V_T > D\}}] &= \mathbb{Q}(V_T > D) \\
&= \mathbb{Q}\left(V_t e^{(r - \frac{1}{2}\sigma_V^2)(T-t) + \sigma_V(\tilde{B}_T - \tilde{B}_t)} > D\right) \\
&= \mathbb{Q}\left(\ln V_t + \left(r - \frac{1}{2}\sigma_V^2\right)(T-t) + \sigma_V(\tilde{B}_T - \tilde{B}_t) > \ln D\right) \\
&= \mathbb{Q}\left(\sigma_V(\tilde{B}_T - \tilde{B}_t) > \ln D - \ln V_t - \left(r - \frac{1}{2}\sigma_V^2\right)(T-t)\right) \\
&= \mathbb{Q}\left(\tilde{y}_1 < \frac{\ln \frac{V_t}{D} + (r - \frac{1}{2}\sigma_V^2)(T-t)}{\sigma_V\sqrt{T-t}}\right)
\end{aligned} \tag{6.25}$$

By expressions (6.24) and (6.25) we get

$$\begin{aligned}
b_1 &= -\frac{\ln \frac{S_t}{X} + (r - \frac{1}{2}\sigma_S^2)(T-t)}{\sigma_S\sqrt{T-t}} \\
b_2 &= \frac{\ln \frac{V_t}{D} + (r - \frac{1}{2}\sigma_V^2)(T-t)}{\sigma_V\sqrt{T-t}}
\end{aligned}$$

Evaluation of term  $\bar{P}_3$ :

$$\begin{aligned}
\bar{P}_3 &= B_t \mathbb{E}_{\mathbb{Q}} [B_T^{-1} X \omega_T I_{\{X > S_T\}} I_{\{V_T < D\}} | \mathcal{F}_t] \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X \frac{V_T}{D} e^{-\frac{1}{2} \sigma_V^2 (T-t) + \sigma_V \tilde{y}_2 \sqrt{T-t}} I_{\{X > S_T\}} I_{\{V_T < D\}} \\
&\quad \frac{1}{2\pi \sqrt{1-\nu^2}} e^{\left(-\frac{1}{2(1-\nu^2)} (\tilde{y}_1^2 - 2\nu \tilde{y}_1 \tilde{y}_2 + \tilde{y}_2^2)\right)} d\tilde{y}_1 d\tilde{y}_2
\end{aligned} \tag{6.26}$$

As in (6.20) we define an equivalent measure, this time with  $\delta_S = \nu \sigma_V$  and  $\delta_V = \delta_S$ . The indicator functions are then evaluated under the new measure,  $\dot{\mathbb{Q}}$ .

$$\begin{aligned}
\mathbb{E}_{\dot{\mathbb{Q}}} [I_{\{X > S_T\}}] &= \dot{\mathbb{Q}}(X > S_T) \\
&= \dot{\mathbb{Q}} \left( X > S_t e^{(r - \frac{1}{2} \sigma_S^2)(T-t) + \sigma_S (\dot{B}_T - \dot{B}_t + \nu \sigma_V (T-t))} \right) \\
&= \dot{\mathbb{Q}} \left( \dot{y}_1 < -\frac{\ln S_t - \ln X + (r - \frac{1}{2} \sigma_S^2 + \nu \sigma_S \sigma_V)(T-t)}{\sigma_S \sqrt{T-t}} \right)
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}_{\dot{\mathbb{Q}}} [I_{\{V_T < D\}}] &= \dot{\mathbb{Q}}(V_T < D) \\
&= \dot{\mathbb{Q}} \left( V_t e^{(r - \frac{1}{2} \sigma_V^2)(T-t) + \sigma_V (\dot{B}_T - \dot{B}_t + \sigma_V (T-t))} < D \right) \\
&= \dot{\mathbb{Q}} \left( \dot{y}_2 < -\frac{\ln V_t - \ln D + (r + \frac{1}{2} \sigma_V^2)(T-t)}{\sigma_V \sqrt{T-t}} \right)
\end{aligned}$$

It follows that

$$\bar{P}_3 = X \frac{V_T}{D} N_2(-d_1, d_2, -\nu)$$

such that

$$\begin{aligned}
d_1 &= -\frac{\ln \frac{S_t}{X} + (r - \frac{1}{2} \sigma_S^2 + \nu \sigma_S \sigma_V)(T-t)}{\sigma_S \sqrt{T-t}} \\
d_2 &= -\frac{\ln \frac{V_t}{D} + (r + \frac{1}{2} \sigma_V^2)(T-t)}{\sigma_V \sqrt{T-t}}
\end{aligned}$$

Evaluation of term  $\bar{P}_4$ :

$$\begin{aligned}
 \bar{P}_4 &= B_t \mathbb{E}_{\mathbb{Q}} \left[ B_T^{-1} S_T \omega_T I_{\{X > S_T\}} I_{\{V_T < D\}} \middle| \mathcal{F}_t \right] \\
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S_t e^{-\frac{1}{2}\sigma_S^2(T-t) + \sigma_S \tilde{y}_1 \sqrt{T-t}} \frac{V_T}{D} e^{(r - \frac{1}{2}\sigma_V^2)(T-t) + \sigma_V \tilde{y}_2 \sqrt{T-t}} \\
 &\quad I_{\{X > S_T\}} I_{\{V_T < D\}} \frac{1}{2\pi\sqrt{1-\nu^2}} e^{\left(-\frac{1}{2(1-\nu^2)}(\tilde{y}_1^2 - 2\nu\tilde{y}_1\tilde{y}_2 + \tilde{y}_2^2)\right)} d\tilde{y}_1 d\tilde{y}_2
 \end{aligned} \tag{6.27}$$

Using (6.18) and setting  $u_1 = \tilde{y}_1 - \sigma_S \sqrt{T-t} - \nu \sigma_V \sqrt{T-t}$  and  $u_2 = \tilde{y}_2 - \sigma_V \sqrt{T-t} - \nu \sigma_S \sqrt{T-t}$  we have

$$\begin{aligned}
 \bar{P}_4 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S_t \frac{V_T}{D} e^{r(T-t)} I_{\{X > S_T\}} I_{\{V_T \geq D\}} \\
 &\quad \frac{1}{2\pi\sqrt{1-\nu^2}} e^{\left(-\frac{1}{2(1-\nu^2)}(u_1^2 - 2\nu u_1 u_2 + u_2^2)\right)} d\tilde{y}_1 d\tilde{y}_2
 \end{aligned}$$

To transform the density function in (6.27) into that of a standard bivariate random variable, we apply a change of measure using the expression as in (6.20) with

$$\begin{aligned}
 \delta_S &= \sigma_S + \nu \sigma_V \\
 \delta_V &= \nu \sigma_V + \nu \sigma_S
 \end{aligned}$$

By expression (6.21) we have

$$\bar{P}_4 = e^{r(T-t)} S_t \frac{V_T}{D} e^{\nu \sigma_S \sigma_V (T-t)} N_2(-c_1, c_2, -\nu)$$

The parameters of the distribution function are again determined by the evaluation of the indicator functions. Evaluating the indicator functions gives

$$\begin{aligned}
 \mathbb{E}_{\dot{\mathbb{Q}}}[I_{\{X > S_T\}}] &= \dot{\mathbb{Q}}(X > S_T) \\
 &= \dot{\mathbb{Q}}\left(X > S_t e^{(r - \frac{1}{2}\sigma_S^2)(T-t) + \sigma_S(\dot{B}_T - \dot{B}_t + (\sigma_S + \nu \sigma_V)(T-t))}\right) \\
 &= \dot{\mathbb{Q}}\left(\sigma_S(\dot{B}_T - \dot{B}_t) > \ln S_t - \ln X + \left(r + \frac{1}{2}\sigma_S^2 + \nu \sigma_S \sigma_V\right)(T-t)\right) \\
 &= \dot{\mathbb{Q}}\left(\dot{y}_1 < -\frac{\ln S_t - \ln X + \left(r + \frac{1}{2}\sigma_S^2 + \nu \sigma_S \sigma_V\right)(T-t)}{\sigma_S \sqrt{T-t}}\right)
 \end{aligned} \tag{6.28}$$



$$\begin{aligned}
\mathbb{E}_{\dot{\mathbb{Q}}}[I_{\{V_T < D\}}] &= \dot{\mathbb{Q}}(V_T < D) \\
&= \dot{\mathbb{Q}}\left(V_t e^{(r - \frac{1}{2}\sigma_V^2)(T-t) + \sigma_V(\dot{B}_T - \dot{B}_t + (\sigma_V + \nu\sigma_S)(T-t))} < D\right) \\
&= \dot{\mathbb{Q}}\left(\sigma_V(\dot{B}_T - \dot{B}_t) < \ln D - \ln V_t - \left(r + \frac{1}{2}\sigma_V^2 + \nu\sigma_S\sigma_V\right)(T-t)\right) \\
&= \dot{\mathbb{Q}}\left(\dot{y}_2 < -\frac{\ln V_t - \ln D + \left(r + \frac{1}{2}\sigma_V^2 + \nu\sigma_S\sigma_V\right)(T-t)}{\sigma_V\sqrt{T-t}}\right)
\end{aligned} \tag{6.29}$$

From the second equality in expression (6.24) and from expressions (6.28) and (6.29), it follows that

$$\bar{P}_4 = e^{r(T-t)} S_t \frac{V_t}{D} e^{\nu\sigma_S\sigma_V(T-t)} N_2(-c_1, c_2, -\nu)$$

where

$$\begin{aligned}
c_1 &= -\frac{\ln \frac{S_t}{X} + \left(r + \frac{1}{2}\sigma_S^2 + \nu\sigma_S\sigma_V\right)(T-t)}{\sigma_S\sqrt{T-t}} \\
c_2 &= -\frac{\ln \frac{V_t}{D} + \left(r + \frac{1}{2}\sigma_V^2 + \nu\sigma_S\sigma_V\right)(T-t)}{\sigma_V\sqrt{T-t}}
\end{aligned}$$

This completes the proof of the Proposition.

$X$	0.000001	20	40	60	80	100	120	140	160	180	200
$\bar{P}$	0	0	0	0.1	1.38	6.03	15.06	27.60	42.23	57.93	74.10
$S$	0.000001	20	40	60	80	100	120	140	160	180	200
$\bar{P}$	82.56	64.31	46.07	28.43	14.94	6.03	2.23	0.77	0.25	0.08	0.03
$S$	0.000001	20	40	60	80	100	120	140	160	180	200
$\sigma_S = 0.1$	82.56	64.31	46.06	27.81	10.73	1.73	0.12	0.00	0.00	0.00	0.00
$\sigma_S = 0.2$	82.56	64.31	46.07	28.43	14.94	6.03	2.23	0.77	0.25	0.08	0.03
$\sigma_S = 0.4$	82.56	64.33	47.07	32.89	22.50	15.3	10.44	7.17	4.97	3.48	2.46
$S$	0.000001	20	40	60	80	100	120	140	160	180	200
$r = 0.00$	86.46	69.17	51.88	34.85	19.96	9.72	4.18	1.65	0.62	0.22	0.08
$r = 0.05$	82.56	64.31	46.07	28.43	14.94	6.03	2.23	0.77	0.25	0.08	0.03
$r = 0.10$	77.61	58.65	39.72	22.02	9.55	3.41	1.08	0.32	0.09	0.03	0.00
$V$	1	20	40	60	80	100	120	140	170	190	200
$\bar{P}$	1315.04	65.75	32.88	21.91	16.32	12.61	9.79	7.78	6.31	6.05	6.03
$BS$	6.61	6.61	6.61	6.61	6.61	6.61	6.61	6.61	6.61	6.61	6.61
$V_{deep}$	1	20	40	60	80	100	120	140	170	190	200
$\bar{P}$	8077.94	403.9	201.95	134.59	100.24	77.46	60.11	47.81	40.68	37.64	37.05
$BS$	38.06	38.06	38.06	38.06	38.06	38.06	38.06	38.06	38.06	38.06	38.06
$S$	0.000001	20	40	60	80	100	120	140	160	180	200
$T = 1$	88.54	69.92	51.31	32.74	15.81	5.12	1.20	0.22	0.03	0.00	0.00
$T = 2$	82.56	64.31	46.07	28.43	14.94	6.03	2.23	0.77	0.25	0.08	0.03
$T = 3$	77.74	59.68	41.7	25.26	13.28	6.32	2.83	1.23	0.53	0.23	0.10
$S$	0.000001	20	40	60	80	100	120	140	160	180	200
$\sigma_V = 0.1$	88.67	69.07	49.48	30.54	15.40	6.48	2.4	0.82	0.27	0.09	0.03
$\sigma_V = 0.2$	82.56	64.31	46.07	28.43	14.94	6.03	2.23	0.77	0.25	0.08	0.03
$\sigma_V = 0.3$	76.82	59.84	42.87	26.45	13.34	5.61	2.08	0.71	0.23	0.08	0.02
$S$	0.000001	20	40	60	80	100	120	140	160	180	200
$D = 160$	84.12	65.53	46.94	28.97	14.61	6.15	2.28	0.78	0.26	0.08	0.03
$D = 180$	82.56	64.31	46.07	28.43	14.94	6.03	2.23	0.77	0.25	0.08	0.03
$D = 200$	83.87	65.33	46.80	28.88	14.57	6.13	2.27	0.78	0.26	0.08	0.03
$S$	0.000001	20	40	60	80	100	120	140	160	180	200
$\rho = -0.8$	82.56	64.89	47.24	30.23	16.03	4.99	0.35	0.01	0.14	0.10	0.10
$\rho = 0.0$	82.56	64.31	46.07	28.43	14.94	6.03	2.23	0.77	0.25	0.08	0.03
$\rho = 0.8$	82.56	63.83	45.11	26.13	11.95	5.79	2.23	0.83	0.27	0.08	0.03

Table 6.5: Summary of Values of Credit Risky Put Options at varying Parameter Sensitivity

## Chapter 7

# Conclusion

In this dissertation we have proposed techniques and models for the pricing of credit risk and credit risk derivatives. Since the pioneering work of Black and Scholes (1973) [20] and Merton (1974) [94] significant advances have been made in the valuation of credit risk. The first generation credit risk models based on the underlying asset value are being increasingly improved upon with models that have explicit assumptions of the default process. The assumptions usually characterize real world phenomena. The complexity of these models can be used to price derivative instruments that are susceptible to credit risk simultaneously with the credit risk itself. Nonetheless, the general theme of this exposition was to develop models in both the structural and reduced-form model framework and to use this as the basis to price credit risk derivatives as an application of contingent claim analysis.

In chapter 2 we presented an overview of the fundamental concepts and assumptions of financial modelling and valuation techniques we used throughout this dissertation. For example, as a precursor to credit risk modelling we presented an introduction to interest rate modelling and bond markets as fixed income instruments are contingent on interest rates. To make our intuitive concepts precise we developed the technical exposition of the Black-Scholes (1973) [20] option pricing formula both as a merit conceptual framework and instructive passage of the valuation techniques for our credit risk models.

We presented the structural model approach for the pricing of credit risk in chapter 3. In this approach the arrival of the default event is associated with the dynamics of the underlying capital structure of the firm thereby giving an economic interpretation to the consequence of the default event. The concept of structural models originated with Merton (1974) [94] who applied option pricing theory to value claimholders of the firm in terms of derivative contracts. In particular, the equity holders claim is just a European call on the value of the assets of the firm and the bondholders have a right to the par value of the bond, to be received, with the sale of a short put option to the equity holders on the assets of the firm. The primary resolve of the structural framework is to proffer a cogent explanation that default is not an unpredictable event but linked to corporate economic conditions. In this approach we postulated that the evolution of the asset price process is

proxied by the value of traded shares of the firm as an indication of the credit worthiness of the corporate as a borrower and this implicitly suggests that default is not contingent on the historical characteristics of the firm value process. This is the key insight of the structural approach is that the default event is predictable.

The fundamental assumption of the Merton (1974) [94] firm value model is that default can only occur if the value of the assets of the firm is less than the amount of the outstanding debt at some pre-defined time  $T$  in the future. Black and Cox (1976) [18] relaxed this restriction to allow for default to occur instantaneously conditional on the firm value process breaching a lower default threshold. This extension gave rise to a new class of models that is widely known as first-passage time models. Over the years several refinements and additions were made to first-passage time models. Stochastic interest rates and a deviation from the absolute priority rule were the two primary modelling features that emerged from the vast amount of research on structural models. In part, we use these features and efficiency gains on previous models postulated in the Briys and de Varenne (1997) [23] model as a basis to propose a first-passage time model. We show that the stochastic default threshold approach proposed by Briys and de Varenne (1997) [23] adds more generality to the model and has the analytical advantage of producing a closed form solution to the fundamental pricing equation for a default risky bond.

In chapter 4 we presented reduced-form models as the second approach to credit risk analysis. In particular, our focus was on the intensity-based valuation of defaultable claims. In intensity-based models default was modelled as an unpredictable event or, put differently, to have a random time of default. The probability of default of this random event followed a jump process and was therefore associated with an intensity parameter. This intensity parameter was modelled to be constant over time or have a stochastic trajectory. We showed two methods to value defaultable zero coupon bonds. The first method, with the assumption of a constant intensity parameter, modelled the time of default as the time of the first jump of a homogeneous Poisson process. Second, we constructed a Cox process or stochastic intensity process to model the time of default. Under both assumptions we were able to derive identical pricing formulae for default risky bonds.

The second part of this dissertation was devoted to the pricing of credit risk derivatives and we present an exposition on this topic in chapter 6. In the context of the ISDA guidelines credit risk derivatives is a recent industry innovation that writes a derivative contract with credit risk as the underlying. We described the basic structures of three common credit risk derivatives, that is, total return swap, credit default swap and a credit spread option. We showed a reduced form approach for the pricing of a credit spread option. In particular, we present the Das and Sundaram (2000) [34] approach that models credit spreads in the framework of the discrete-time forward rate methodology of the Heath, Jarrow, and Morton (1990) [64] model. We derive a numerical value for the credit spread option. In the last section of this chapter we show the structural model approach under the assumption of constant interest rates and deterministic liabilities can be applied to derive the pricing formula for a credit risky put option.

## 7.1 Practical Aspects of Credit Risk Modelling

In this dissertation the overview of mathematical approaches to credit risk analysis is just a sample of models we can find published. In the main, the models we construct to evaluate credit risk are developed around modelling specified credit events, such as defaults, spread risk and credit migrations, and the payoffs on contingent claims written on these events. Nonetheless, when transitioning between a purely theoretical hypothesis and quantifying empirical regularities it is important to choose the appropriate modelling framework to qualify the analysis at hand. Given that the market we operate in can experience business cycles it would be prudent to recognize that some models may prove to produce qualitative results under particular economic regimes in comparison to different models.

The primary focus of this section is to gain insight on some common practical aspects that affect credit risk modelling and to articulate developments in response to these issues while at the same time discuss the limitations of the mathematical modelling of these innovations. At a practical level the issues that credit risk models are likely to encompass include:

- **Pricing loan obligations and corporate bonds:** These fixed income debt securities have several risks associated with them. Not in the least these risks are represented by fluctuations in the interest rate, risk of defaults, credit rating downgrades and widening of credit spreads. If we defer to the analogy that zero coupon bonds are the basic securities used to construct a term structure of default free bonds, similarly, loans and corporate bonds form the basic building blocks of complex credit risk models.
- **Credit risk management:** This means that our models should be able to adequately quantify the risks associated with a portfolio that has credit risky securities as a component. In addition, this requires monitoring and measuring both expected credit losses and unexpected credit losses. The inherent difficulty of credit risk modelling can be attributed to the fact that credit risk can be characterized by several interdependent variables. As mentioned previously, credit events can be defined as credit downgrades, credit spreads and losses contingent on a default. We observe there exists some degree of correlation among variables that measure the probability of default events and the correlation of the default events themselves. These covariance interactions introduces additional complexity in the computational modelling of credit risk. However, traditional modelling practice tend to specify independent probability distributions for each class of credit risk event.
- **Pricing credit risk derivatives:** In comparison to other financial derivatives the credit risk derivative product is markedly different and standard pricing models cannot be adapted at ease to characterize its nature. In addition, the tractability of some models are implicitly ambiguous. If we restrict ourselves to pricing issues then the

key challenge is to adopt an implied modelling approach, that is, to calibrate the theoretical model to empirical data and to price complex structures off the model.

The sophistication of financial products has largely changed the nature of the way banks and other financial institutions do business. Indeed, to keep apace of these developments institutions have developed new methodologies to assess and manage the credit risk arising from the different aspects of their business processes. For example, to improve internal credit risk management systems financial institutions formulated methodologies into credit risk models that the market has widely come to know as CreditMetrics<sup>TM</sup> (by J.P. Morgan), CreditRisk+<sup>TM</sup> (by Credit Suisse Financial Products), and KMV (by KMV Corporation). These models play a decisive role in active portfolio credit risk management.

From a macro perspective, commercial software packages for credit risk management models the loss distributions incurred by an investor of credit risky securities. The pricing of credit risky securities under a risk-adjusted probability measure is, in general, overlooked. In addition, these approaches are designed to suit portfolio credit risk management and usually rely on simulation methods. These models essentially characterize a new paradigm in credit risk management. While the choices of conceptual methodology that each financial institution adopts to develop a credit risk modelling framework is largely subjective, based on considerations such as the characteristics of the institutions loan portfolio and its particular credit culture, it nonetheless serves to highlight two key shortcomings in each credit risk model.

1. Data limitations: Both institutions and researchers alike record a lack of useful data places restrictive assumptions on the design and implementation of robust credit risk models. The lack of a comprehensive record of historical prices required to estimate credit risk in models arises from the stochastic nature of credit default events and the arbitrary if not longer time horizons used in measuring credit risk. Hence, present credit risk models rely on simplifying structural assumptions and parameter estimates derived from proxy data.
2. Model validation: The validation and back-testing of credit risk models is fundamentally difficult since those models rely on a time horizon of one year or more. The longer time horizon, coupled with stricter confidence intervals used in credit risk models creates inherent problems in designing and assessing the accuracy of these models while quantitative validation would require an impractical number of years of data, spanning multiple credit cycles.

The portfolio credit risk models may by design be influenced by changes in credit quality, market variables and credit cycles and consequently the modelling methodology holds out the scenario for a more responsive and informative tool for credit risk management. However, some of the potential benefits of the above set of credit risk models include:

- The credit risk models assess and provide estimates of credit risk, which reflect singular portfolio consumption and a such may provide better insight of concentration risk compared to non portfolio approaches.

- A financial institution's initial credit risk exposures typically cut across sectors and product lines and the use of credit risk models creates a framework to examine this risk in a timely manner and analyze marginal and absolute concentrations to risk.
- In addition, credit risk models may offer: (a) the incentive to improve the institutions overall ability to identify, measure and manage risk (b) improve systems and data collection efforts and (c) provide a more accurate and performance based approach to pricing portfolio credit risk.

The CreditMetrics<sup>TM</sup>, CreditRisk+<sup>TM</sup>, and KMV model can be considered as reasonable models to assess portfolio credit risk for pure loans and bonds. All the models assume deterministic interest rates and are therefore inappropriate to measure credit risk for non linear derivative instruments like swaps or credit linked notes. Indeed, the next generation credit risk models need to propose a framework that at least allows for stochastic interest rates and integrates, in a consistent manner, both credit exposure and loss distribution. Nonetheless, the commercial credit risk models serves as an innovative approach to advances in portfolio credit risk management in that they achieve a desirable alternative to classical credit risk management.

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